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Lyapunov Exponents

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TRIESTE LECTURE NOTES ON LYAPUNOV EXPONENTS PART I

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Abstract. These are the lecture notes for the first week of the course "Lyapunov exponents" at the School and Workshop on Dynamical Systems, ICTP (Trieste, Italy), July 2008. In order to explain interesting results keeping a low level of technicalities, we restrict ourselves to $\mathrm{SL}(2,\mathbb{R})$ cocycles.

Contents

1. $\mathrm{SL}(2,\mathbb{R})$-Cocycles and Their Lyapunov Exponents

1.1. **Cocycles.** Let $T : X \to X$ be a discrete dynamical system. A *skew product* over T is a dynamical system F acting on a product space $X \times Y$ such that $\pi \circ F = T$, where $\pi : X \times Y \to X$ is the projection on the first coordinate. We are interested in skew-products that act *linearly* on the second coordinate – so Y needs to be a vector space. Such skew products are called *linear cocycles*.

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In these notes we will focus in a even more restricted class of examples. Let $\mathrm{SL}(2,\mathbb{R})$ be the group of real 2×2 matrices with determinant 1. A $\mathrm{SL}(2,\mathbb{R})$-*cocycle* over the dynamical system $T : X \to X$ is a map

$$F\colon (x,v) \in X\times\mathbb{R}^2 \mapsto (Tx, A(x)\cdot v) \in X\times\mathbb{R}^2, \quad \text{where} \quad A : X \to \mathrm{SL}(2,\mathbb{R})\,.$$

Since the pair (T,A) specifies $F = F_{T,A}$, we also call it a cocycle. Sometimes (when the underlying T is fixed) we also call the map A a cocycle.

The powers F^n can be written as $F^n(x,v) = (T^n x, A^n_T(x)\cdot v)$, where

$$A^n_T(x) = A(T^{n-1}x)\cdots A(x) \quad (\text{for } n>0).$$

In the case T is invertible, so is F, so we can define

$$A^{-n}_T(x) = \left[A(T^{-n}x)\cdots A(T^{-1}x)\right]^{-1} \quad (\text{for } n>0), \quad A^0_T(x) = Id.$$

Thus the following "cocyle identity"

$$A^{m+n}_T(x) = A^m_T(T^n x)\, A^n_T(x) \quad \text{for all } x\in X,\ m,n\in\mathbb{Z}. \tag{1.1}$$

Most of the time T will be fixed and we write for simplicity $A^n(x)$ instead of $A^n_T(x)$.

Note. Of course these concepts can be generalized in several ways. One can define similarly linear cocycles on vector bundles. Also, cocycles with continuous time can be defined by means of an identity analogous to (1.1), with $\mathbb{Z}$ replaced by $\mathbb{R}$. See [2].

We will deal here with ergodic theory of $\mathrm{SL}(2,\mathbb{R})$-cocycles.[1] Thus we let $(X,\mathcal{A},\mu)$ be a probability space; then we only deal with cocycles (T,A) such that T preserves the measure μ and $A : X\to \mathrm{SL}(2,\mathbb{R})$ is measurable. (Sometimes we consider more regular classes of maps: continuous, differentiable etc.)

Let us mention some situations where these cocycles appear:

Example 1.1. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. Given a diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ that preserves area and orientation, we can define a $\mathrm{SL}(2,\mathbb{R})$-cocycle (T,A) by taking $X = \mathbb{T}^2$, μ as Lebesgue measure, $T = f$, and A as the derivative of f (because the tangent bundle is trivial, i.e., $TX = X\times\mathbb{R}^2$).

Example 1.2. Let η be a Borel probability measure on $\mathrm{SL}(2,\mathbb{R})$. Let Y_0, Y_1, … be a sequence of independent random matrices, identically distributed according to η. To study the behavior of the random

[1] In fact, in most of these notes $\mathrm{SL}(2,\mathbb{R})$ could have been replaced with the group of real 2×2 matrices with determinant ± 1.

products $Y_{n-1}\cdots Y_1 Y_0$, it is convenient to consider a cocycle as follows: Let X be the space $\mathrm{SL}(2,\mathbb{R})^{\mathbb{N}}$ of (one-sided) sequences of matrices $(M_0, M_1, \ldots)$, endowed with the probability measure $\mu = \eta^{\mathbb{N}}$. The shift $T : X \to X$ defined by $T(M_0, M_1, \ldots) = (M_1, M_2, \ldots)$ preserves the measure μ (and is ergodic with respect to it). Let $A : X \to \mathrm{SL}(2,\mathbb{R})$ be the map $(M_0, M_1, \ldots) \mapsto M_0$. Then if x is a random point in X, $A^n_T(x)$ is a random product of n i.i.d. matrices.

1.2. **The Lyapunov Exponent.** Given a cocycle $F = F_{T,A}$, we want to understand the behavior of typical orbits $F^n(x, v) = (T^n x, A^n(x)\cdot v)$, as it is usual in Ergodic Theory. Thus we aim to obtain information about the sequence of matrices $A^n(x)$, for a full-measure set of points x. The most basic information of this kind concerns asymptotic growth.

Let $\|\cdot\|$ indicate the euclidian norm on $\mathbb{R}^2$ and also the induced operator norm on the space $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ of linear maps (that is, $\|L\| = \sup_{\|v\|=1} \|Lv\|$). If $L \in \mathrm{SL}(2,\mathbb{R})$ then $\|L\| = \|L^{-1}\| \geq 1$.

Theorem 1.3 (Furstenberg–Kesten Theorem [10] for $\mathrm{SL}(2,\mathbb{R})$)**.** *Assume $T : X \to X$ is a μ-preserving transformation and $A : X \to \mathrm{SL}(2,\mathbb{R})$ is a measurable map such that:*

$$\int \log \|A(x)\|\, d\mu(x) < \infty. \tag{1.2}$$

Then for μ-almost every $x \in X$, the following limit exists:

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|. \tag{1.3}$$

The function $\lambda : X \to [0,\infty)$ is T-invariant, μ-integrable, and its integral is given by

$$\Lambda = \int \lambda\, d\mu = \lim_{n\to\infty} \frac{1}{n} \int \log \|A^n\|\, d\mu = \inf_{n\geq 1} \frac{1}{n} \int \log \|A^n\|\, d\mu\,. \tag{1.4}$$

We call (1.2) the *integrability condition*. The number $\lambda(x)$ is called the *(upper) Lyapunov exponent* at the point x, and Λ is called the *integrated (upper) Lyapunov exponent*. If T is ergodic then λ is constant equal to Λ almost everywher, so we write $\lambda = \Lambda$ for simplicity.

Also notice that any norm on $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ would work equally well in the statement of Theorem 1.3

The proof of Theorem 1.3 will be based on the following result (see e.g. [13] for a proof):

Theorem 1.4 (Kingman's Subadditive Ergodic Theorem [14])**.** *Let $f_n : X \to \bar{\mathbb{R}}$ be a sequence of measurable functions such that f_1^+ is μ-integrable* *and*

$$f_{m+n} \leq f_m + f_n \circ T^m \quad \textit{for all } m, n \geq 1.$$

Then $\frac{1}{n} f_n$ *converges a.e. to a function* $f : X \to \bar{\mathbb{R}}$*. Moreover,* f^+ *is* μ*-integrable and*

$$\int f = \lim_{n\to\infty} \frac{1}{n} \int f_n = \inf_{n\geq 1} \frac{1}{n} \int f_n \in \mathbb{R} \cup \{-\infty\}$$

A sequence of functions as in the hypotheses of the theorem is called *subadditive*.

Proof. Proof of Theorem 1.3 The sequence of functions $f_n(x) = \log \|A^n(x)\|$ is subadditive, and f_0 is integrable. Therefore Theorem 1.4 assures that f_n/n converges almost everywhere to a function λ. Since $f_n \geq 0$, $\lambda \geq 0$. Theorem 1.4 also gives (1.4). □

Note. For Theorem 1.3 in higher dimension, see [2].

1.3. **Oseledets Theorem.**

1.3.1. *Statement.* Theorem 1.3 gives information about the growth of the matrices $A^n(x)$, while the Oseledets [16] Theorems below describe the asymptotic behavior of vectors $A^n(x) \cdot v$.

Theorem 1.5 (One–Sided Oseledets)**.** *Let* $T : X \to X$ *be a* μ*-preserving transformation and* $A : X \to \mathrm{SL}(2, \mathbb{R})$ *satisfy the integrability condition* (1.2)*. Let* $\lambda(\cdot)$ *be the Lyapunov exponent of the cocycle* (T, A)*. If* $\lambda(x) = 0$ *then*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = 0 \quad \textit{for every } v \in \mathbb{R}^2 \smallsetminus \{0\}. \tag{1.5}$$

For a.e. $x \in X$ *such that* $\lambda(x) > 0$*, there exists a one-dimensional vector space* $E_x^- \subset \mathbb{R}^2$ *such that:*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \begin{cases} \lambda & \textit{for all } v \in \mathbb{R}^2 \smallsetminus E_x^- \\ -\lambda & \textit{for all } v \in E_x^- \smallsetminus \{0\} \end{cases} \tag{1.6}$$

Moreover, the spaces E_x^- *depend measurably on the point* x *and are invariant by the cocycle.*

Measurability of the spaces E_x^- means that they give a measurable map from the set $\{x;\ \lambda(x) > 0\}$ to $\mathbb{P}^1$ (the projective space of $\mathbb{R}^2$), while invariance means that $A(x) \cdot E_x^- = E_{Tx}^-$.

Thus Theorem 1.5 says that if $\lambda(x) > 0$ then $\|A^n(x) \cdot v\|$ grows like $e^{n\lambda(x)}$ for v in all directions in $\mathbb{R}^2$, except for one direction for which the growth is like $e^{-n\lambda(x)}$.

For invertible cocycles we have:

Theorem 1.6 (Two-sided Oseledets)**.** *Let T be an invertible bimeasurable transformation of the probability space (X, μ), and let $A : X \to \mathrm{SL}(d, \mathbb{R})$ satisfy the integrability condition. For a.e. point x where the Lyapunov exponents is positive, there exists a splitting $\mathbb{R}^2 = E^+_x \oplus E^-_x$ into two linear one-dimensional subspaces such that* (1.6) *holds and*

$$\lim_{n\to-\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \begin{cases} \lambda & \text{for all } v \in \mathbb{R}^2 \setminus E^+_x \\ -\lambda & \text{for all } v \in E^+_x \setminus \{0\} \end{cases} \tag{1.7}$$

Moreover, the spaces E^+_x and E^-_x are invariant by the cocycle, depend measurably on the point x, and satisfy:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \measuredangle\left(E^+_{T^n x}, E^-_{T^n x}\right) = 0\,. \tag{1.8}$$

We call E^+_x and E^-_x the *Oseledets spaces*. In view of (1.8), we say that the angle between them is *subexponential*.

Note. For more general versions of Oseledets Theorem, see [2].

1.3.2. *Proof of Oseledets Theorem.* Any matrix $A \in \mathrm{SL}(2, R)$ can be written as $A = RDS$, where R and S belong to $\mathrm{SO}(2, \mathbb{R})$ (the group of rotations), and D is diagonal with non-negative entries. Notice than $\|A\| = \|D\|$. It follows that if A is not a rotation, then there are unit vectors $u(A) \perp s(A)$ unique modulo sign, such that

$$\|A \cdot u(A)\| = \|A\|, \quad \|A \cdot s(A)\| = \|A\|^{-1}\,.$$

Moreover $A(u(A))$ and $A(s(A))$ are collinear respectively to $s(A^{-1})$ and $u(A^{-1})$. Notice $u(A)$ and $s(A)$ are the eigenvectors of the symmetric matrix A^*A.

The following lemma will be used a few times:

Lemma 1.7. *Let $f : X \to \mathbb{R}$ be a measurable function such that $f \circ T - f$ is integrable in the extended sense*[2]*. Then*

$$\lim_{n\to\infty} \frac{1}{n} f(T^n x) = 0 \quad \text{for a.e. } x \in X.$$

Proof. Let $g = f \circ T - f$, and assume $g^+ \in L^1(\mu)$. By Birkhoff's Theorem, there is a function $\tilde{g}$ with $\tilde{g}^+ \in L^1(\mu)$ such that

$$\frac{f \circ T^n}{n} = \frac{f}{n} + \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j \to \tilde{g} \text{ a.e.}$$

[2]A measurable function g is said to be integrable in the extended sense if g^+ or g^- are integrable. Notice that the Birkhoff Theorem still applies.

For every point x where convergence above holds and $\tilde{g}(x) \neq 0$, we have $|f(T^n x)| \to \infty$. But, by the Poincaré's Recurrence Theorem, the set of points x which satisfy the latter condition has zero measure. Therefore $\tilde{g} = 0$ a.e., as we wanted to show. □

Proof of Theorem 1.5. Let $\lambda(\cdot)$ be given by Theorem 1.3. For each point such that (1.3) holds and $\lambda(x) = 0$. Then, for every non-zero $v \in \mathbb{R}^2$,

$$\|A^n(x)\|^{-1} \|v\| \le \|A^n(x) \cdot v\| \le \|A^n(x)\| \|v\| .$$

Taking log's, dividing by n, and making $n \to +\infty$ gives (1.5).

Now consider the T-invariant set $[\lambda > 0] = \{x \in X;\ \lambda(x) > 0\}$. For a.e. $x \in [\lambda > 0]$, the orthogonal directions $s_n(x) = s(A^n(x))$, $u_n(x) = u(A^n(x))$ are defined for sufficiently large n. We are going to show that they converge to (necessarily measurable) maps $[\lambda > 0] \to \mathbb{P}^1$, and that $\lim s_n(x)$ is exactly the E^-_x space we are looking for.

Fix some x with $\lambda(x) > 0$. We may write λ, s_n instead of $\lambda(x)$, $s_n(x)$ etc. Take unit vectors in the directions of s_n and u_n that by simplicity of notation we indicate by the same symbols.

Let $\alpha_n > 0$ be the angle between s_n and s_{n+1}. That is, $s_n = \pm\cos\alpha_n\, s_{n+1} \pm \sin\alpha_n\, u_{n+1}$. Since the vectors s_{n+1}, u_{n+1} are orthogonal and so are their images by $A^{n+1}(x)$, we get:

$$\|A^{n+1}(x) \cdot s_n\| \ge \|A^{n+1}(x) \cdot (\sin\alpha_n\, u_{n+1})\| = (\sin\alpha_n)\|A^{n+1}(x)\| .$$

On the other hand:

$$\|A^{n+1}(x) \cdot s_n\| \le \|A(T^n x)\| \, \|A^n(x) s_n\| = \|A(T^n x)\| \, \|A^n(x)\|^{-1} .$$

So it follows that

$$\sin\alpha_n(x) \le \frac{\|A(T^n x)\|}{\|A^n(x)\| \, \|A^{n+1}(x)\|} . \tag{1.9}$$

From the definition (1.3) of λ, the integrability condition (1.2), and Lemma 1.7 it follows that for a.e. x,

$$\lim_{n\to\infty} \frac{1}{n} \log\sin\alpha_n = -2\lambda .$$

Thus $\alpha_n(x)$ goes exponentially fast to zero, and, in particular, $s_n(x)$ is a Cauchy sequence in $\mathbb{P}^1$, for a.e. x. Let $s(x)$ be the limit. As the tail of a geometric series goes to zero with the same speed as the summand, we have:

$$\lim_{n\to\infty} \frac{1}{n} \log\sin\measuredangle(s_n, s) = -2\lambda . \tag{1.10}$$

Now write $\beta_n = \measuredangle(s_n, s)$. Then

$$A^n(x) \cdot s = \pm\|A^n(x)\|^{-1} \cos\beta_n \pm \|A^n(x)\| \sin\beta_n$$

Therefore, using (1.10),

$$\limsup_{n\to\infty}\frac{1}{n}\log\|A^n(x)\cdot s\| \le \limsup_{n\to\infty}\frac{1}{n}\log\max\left(\|A^n(x)\|^{-1},\|A^n(x)\|\sin\beta_n\right)$$
$$= \max(-\lambda,\lambda-2\lambda) = -\lambda\,.$$

On the other hand, $\frac{1}{n}\log\|A^n(x)\cdot s\| \ge \frac{1}{n}\log\|A^n(x)\|^{-1} \to -\lambda$, so it follows that $\frac{1}{n}\log\|A^n(x)\cdot s\| \to -\lambda$. Now, if v is a unit vector not collinear to s then

$$\|A^n(x)\cdot v\| \ge \|A^n(x)\|\sin\measuredangle(v,s_n),$$

which implies that $\frac{1}{n}\log\|A^n(x)\cdot v\| \to \lambda$. So we have proved that (1.6) holds taking E^-_x as the s direction. Finally, notice that if $v\in A(x)\cdot E^-_x\smallsetminus\{0\}$ then $\frac{1}{n}\log\|A^n(Tx)\cdot v\|\to-\lambda$. It follows that $v\in E^-_{Tx}$ almost surely. So invariance holds and the proof of Theorem 1.5 is completed. □

We now consider the invertible case:

Proof of Theorem 1.6. Let E^- and E^+ be the spaces given by Theorem 1.5 applied respectively to $F=F_{T,A}$ and F^{-1}. Then (1.6) and (1.7) hold. To show that $E^-_x\neq E^+_x$ a.e., it is sufficient to show that:

$$\lim_{n\to+\infty}\frac{1}{n}\log\|A^{-n}(x)|E^-_x\| = -\lambda(x) \quad \text{for a.e. } x\in[\lambda>0] \tag{1.11}$$

By (1.7), the limit on the left hand side exists a.e. and defines a measurable function $g(x)$. For all $n\ge 0$ and $x\in[\lambda>0]$, we have

$$A^{-n}(x)|E^-_x = \left(A^n(T^{-n}x)|E^-_{T^{-n}x}\right)^{-1},$$

Thus $g_n = f_n\circ T^{-n}$, where $f_n=\frac{1}{n}\log\|A^n|E^-\|$ and $g_n=\frac{1}{n}\log\|A^{-n}|E^-\|$. By (1.6), f_n converges a.e. to $-\lambda$. Since λ is T-invariant, for all $\varepsilon>0$, $\mu[|g_n+\lambda|>\varepsilon]=\mu[|f_n+\lambda|>\varepsilon]\to 0$. That is, $g_n\to-\lambda$ in measure. Since $g_n\to g$ a.e., we conclude that $g=-\lambda$ a.e., proving (1.11).

We are left to prove (1.8). It is easy to see that for any $L\in\mathrm{SL}(2,\mathbb{R})$ and any non-zero vectors $v,w\in\mathbb{R}^2$,

$$\|L\|^{-2}\le\frac{\sin\measuredangle(Lv,Lw)}{\sin\measuredangle(v,w)}\le\|L\|^2\,.$$

Thus if $\phi=\log\sin\measuredangle(E^+,E^-)$ then $|\phi\circ T-\phi|\le 2\log\|A\|$. By the integrability condition and Lemma 1.7 we conclude that $\frac{1}{n}\phi\circ T^n\to 0$, as we wanted to show. □

1.4. **Conjugacy.** For a fixed $T : X \to X$, we say a measurable map $C : X \to \mathrm{SL}(2,\mathbb{R})$ is *tempered* if $\lim_{n\to\infty} \frac{1}{n} \log \|C^{\pm 1}(T^n x)\| = 0$ for a.e. x.

We say that two cocycles $A, B : X \to \mathrm{SL}(2,\mathbb{R})$ (over the same base dynamics T) are *conjugate* (or *cohomologous*) if there is a tempered $C : X \to \mathrm{SL}(2,\mathbb{R})$ such that

$$B(x) = C(Tx)^{-1}A(x)C(x)$$

That is, $F_{T,B} = F_{C,id}^{-1} \circ F_{T,A} \circ F_{C,id}$.

It is evident that if the Lyapunov exponent is a conjugacy invariant among $\mathrm{SL}(2,\mathbb{R})$-cocycles. (Assume for simplicity that T is invertible ergodic.) If the Lyapunov exponent of a certain A is positive then using Oseledets Theorem (more specifically, property 1.8), we can find a tempered conjugacy between A and a cocycle formed by diagonal matrices. The conjugacy classes of cocycles with zero Lyapunov exponent were studied by Thieullen [18]. He shows they fall in three cases: elliptic, parabolic, and "flip"; we will see them in the forthcoming Examples 3.1, 3.2, and 3.3, respectively.

Using a conjugacy with a diagonal cocycle, it is easy to prove the following property about angles:

$$v \in \mathbb{R}^2 \setminus E_x^- \;\Rightarrow\; \lim_{n\to\infty} \frac{1}{n} \log \measuredangle(A^n(x)\cdot v, E^+_{T^n x}) = -2\lambda(x). \tag{1.12}$$

2. Examples

2.1. **Uniform Hyperbolicity.** A whole class of examples that deserves to be studied in some detail is that of the uniformly hyperbolic cocycles.

In this subsection we assume X is a compact Hausdorff space.

Let $T : X \to X$ and $A : X \to \mathrm{SL}(2,\mathbb{R})$ be continuous maps. We say the cocycle in *uniformly hyperbolic* if there exist constants $c > 0$ and $\tau > 0$ such that

$$\|A^n(x)\| > ce^{\tau n}, \quad \text{for all } n \ge 0. \tag{2.1}$$

Note. Our definition of uniform hyperbolicity is apparently weaker than the more usual one; but we will establish their equivalence in Theorem 2.3 and Corollary 2.5 below.

Remark 2.1. For any $k \ge 1$, a cocycle (T, A) is uniformly hyperbolic if and only if so is its power (T^k, A_T^k).

Example 2.2 ([20])**.** Assume (T, A) is a measurable $\mathrm{SL}(2,\mathbb{R})$-cocycle where all matrices $A(x)$ have positive entries. Let us prove that $\lambda > 0$.

Notice that:

$$a, b, c, d, v_1, v_2 > 0,\ ad-bc = 1,\ \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v_1' v_2' > (1+2bc) v_1 v_2 .$$

Thus if $v = (v_1, v_2)$ has positive entries then the product of the entries of $A^n(x) \cdot v$ is at least $v_1 v_2 \prod_{i=0}^{n-1} (1 + 2b(T^i x)c(T^i x))$. This number grows exponentially as $n \to \infty$ (by Birkhoff's Theorem), and hence so does $\|A^n(x) \cdot v\|$. This shows that $\lambda > 0$. The argument also shows that if the space is compact and the cocycle is continuous, so that the matrix entries are bounded away from 0, then the cocycle is uniformly hyperbolic.

Theorem 2.3 (Prop. 2 in [19]). *If (T, A) is a uniformly hyperbolic cocycle then there exist a map $E^s : X \to \mathbb{P}^1$ and constants $C > 0$ and $\sigma > 0$ such that*

$$\|A^n(x)|E^s_x\| < Ce^{-\sigma n}, \quad \textit{for all } x \in X \textit{ and } n \geq 0. \tag{2.2}$$

Moreover, the map E^s is unique, invariant by the cocycle, and continuous.

Proof. Assume (T, A) is uniformly hyperbolic, and fix $\tau > 0$ so that (2.1) is satisfied. We will use Oseledets Theorem 1.5 and its proof. Let $s_n(x)$ be the direction the most contracted by $A^n(x)$. We have the estimate (1.9) for the angle $\alpha_n(x) = \measuredangle(s_n(x), s_{n+1}(x))$. Here A is uniformly bounded, so we obtain from (2.1) that $\alpha_n(x)$ goes exponentially fast to zero, uniformly in x. In particular, $E^s(x) = \lim s_n(x)$ exists and is a continuous function.

We want to prove that E^s is uniformly contracted. The matrix calculations in the proof of Theorem 1.5 do not seem to give this, so we will use an ergodic-theoretic argument. Let μ be any T-invariant Borel probability measure. By the proof of Theorem 1.5, we know that $E^s(x)$ is the Oseledets contracting direction for μ-a.e. $x \in X$. Consider the continuous function $\phi(x) = \log \|A(x)|E^s(x)\|$. Its n-th Birkhoff average is

$$B_n(x) = \frac{1}{n}(\phi + \phi \circ T + \cdots + \phi \circ T^{n-1}) = \frac{1}{n} \log \|A^n(x)|E^s(x)\| .$$

By Oseledets' Theorem, for μ-a.e. $x \in X$, $\lim B_n(x)$ exists and equals $-\lambda(x) = -\lim \frac{1}{n} \log \|A^n(x)\|$. By the hypothesis (2.1), $\lambda(x) \geq \tau$. Now we need the following:

Lemma 2.4. *Let $\phi : X \to \mathbb{R}$ be a continuous function, and let B_n denote the n-Birkhoff average of ϕ under T. Assume that there is $a \in \mathbb{R}$ such that for every T-invariant measure μ, we have $\lim_{n\to\infty} B_n(x) \leq a$ for μ-a.e.*

$x \in X$. *Then* $\limsup_{n\to\infty} B_n(x) \le a$ uniformly. *That is, for every* $a' > a$ *there exists* $n_0 \in \mathbb{N}$ *such that* $B_n(x) < a'$ *for every* $n \ge n_0$ *and every* $x \in X$.

Proof. This is a standard Krylov–Bogoliubov argument. If the conclusion is false then there is $a' > a$ and sequences $n_i \to \infty$ and $x_i \in X$ such that $B_{n_i}(x_i) \ge a'$. Consider the sequence of measures $\mu_i = \frac{1}{n_i}\sum_{j=0}^{n_i-1} \delta_{T^j x_i}$. Passing to a subsequence, we can assume that μ_i converges weakly to a measure μ. Then μ is T-invariant and $\int \phi\, d\mu = \lim \int \phi\, d\mu_i = \lim B_{n_i}(x_i) \ge a'$. So by Birkhoff's Theorem, the set of points x such that $\lim B_n(x) \ge a'$ has positive μ measure. This contradicts the assumption. □

Coming back to the proof of Theorem 2.3, it follows from the lemma that $\limsup_{n\to\infty} B_n(x) \le -\tau$ uniformly. In particular, for any $\sigma < \tau$, there exist n_0 such that $B_{n_0}(x) < -\sigma$ for every $x \in X$. Thus (by the same argument as for Remark 2.1), (2.2) holds for appropriate C.

Let us show uniqueness of E^s. If for some x there existed two linearly independent vectors v_1, v_2 in $\mathbb{R}^2$ such that $\lim_{n\to\infty} A^n(x)\cdot v_i = 0$ for both $i = 1, 2$ then we would have $\|A^n(x)\| \to 0$, which is impossible. The invariance of E^s is a consequence of uniqueness. □

Corollary 2.5. *If* $T : X \to X$ *is a homeomorphism and* (T, A) *is uniformly hyperbolic then there is a continuous invariant splitting* $\mathbb{R}^2 = E^u_x \oplus E^s_x$ *such that*

$$\|A^{-n}(x)|E^u_x\| < Ce^{-\sigma n}, \quad \|A^n(x)|E^s_x\| < Ce^{-\sigma n}, \quad \text{for all } x \in X \text{ and } n \ge 0,$$

where $C > 0$ *and* $\sigma > 0$ *are constants. The spaces* E^u_x *and* E^s_x *are uniquely defined and are invariant by the cocycle.*

Proof. Let E^s and E^u be given by Theorem 2.3 applied respectively to the cocycle and its inverse. Since $\|A^{-n}(x)|E^s_x\| = \|A^n(T^{-n}x)|E^s_{T^{-n}x}\|^{-1} \to \infty$ as $n \to +\infty$, we see that $E^s_x \ne E^u_x$. □

The spaces E^u and E^s are called respectively the *unstable* and *stable* directions. By continuity, the angle between them has a positive lower bound.

Proposition 2.6. *Let* (T, A) *be a uniformly hyperbolic cocycle. Then for every continuous map* $B : X \to \mathrm{SL}(2, \mathbb{R})$ *sufficiently close to* A*, the cocycle* (T, B) *is uniformly hyperbolic.*

Proof. Let $E^s : X \to \mathbb{P}^1$ be the stable direction. For $\alpha > 0$, define the following *cone field*:

$$C^s_\alpha(x) = \left\{ v \in \mathbb{R}^2;\ \measuredangle(v, E^s_x) \le \alpha \text{ or } v = 0 \right\}.$$

It is easy to see that there is α and $k \geq 1$ such that for every $x \in X$,

$$v \in C_\alpha(T^k(x)), \quad w = [A^k(x)]^{-1} \cdot v \quad \Rightarrow \quad \begin{cases} w \in C_{\alpha/2}(x) \\ \|w\| > 2\|v\| \end{cases}$$

Therefore if B is sufficiently close to A then

$$v \in C_\alpha(T^k(x)), \quad w = [B^k(x)]^{-1} \cdot v \quad \Rightarrow \quad \begin{cases} w \in C_{\alpha}(x) \\ \|w\| > 2\|v\| \end{cases}$$

It follows that for any $m \geq 1$ and $v \in C_\alpha(T^{km}x)$ we have $\|[B^{km}(x)]^{-1} \cdot v\| > 2^m\|v\|$. So $\|B^{km}(x)\| > 2^m$. This proves that (T^k, B^k_T) is uniformly hyperbolic, and thus by Remark 2.1, so is (T, B). $\square$

Example 2.7. Let $T : X \to X$ be a homeomorphism. Let $f : X \to \mathbb{R}$ be a continuous positive function, and define diagonal matrices $B(x) = \exp(f(x)Id)$. For any continuous $C : X \to \mathrm{SL}(2, \mathbb{R})$, the cocycle (T, A) with $A(x) = C(Tx)^{-1}\, B(x)\, C(x)$ is uniformly hyperbolic. However, it is not true that all uniformly hyperbolic cocycles (T, A) are of this form, because topological obstructions may arise.

The existence of the continuous invariant direction imposes topological obstructions to uniform hyperbolicity:

Example 2.8. Let X be the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Assume (T, A) be a uniformly hyperbolic cocycle with stable direction E^s. There exist integers[3] $\mathbf{t}$, $\mathbf{a}$, $\mathbf{e}$ such that the maps

$$T : \mathbb{T}^1 \to \mathbb{T}^1\,, \quad A : \mathbb{T}^1 \to \mathrm{SL}(2, \mathbb{R})\,, \quad E^s : \mathbb{T}^1 \to \mathbb{P}^1$$

are respectively homotopic to

$$x \mapsto \mathbf{t}x\,, \quad x \mapsto R_{2\pi\mathbf{a}x}\,, \quad x \mapsto \mathbb{R}(\cos \pi\mathbf{e}x, \sin \pi\mathbf{e}x)\,.$$

(R_θ denotes the rotation of angle θ.) Then the invariance relation $A(x) \cdot E^s(x) = E^s(Tx)$ implies that $2\mathbf{a} + \mathbf{e} = \mathbf{te}$, that is,

$$2\mathbf{a} = (\mathbf{t} - 1)\mathbf{e}\,. \tag{2.3}$$

Now, for some values of $\mathbf{t}$ and $\mathbf{a}$, the equation above has no integer solution $\mathbf{e}$. Therefore there exist many homotopy classes $\mathcal{U}(\mathbf{t}, \mathbf{a})$ that contain no uniformly hyperbolic cocycle (T, A).

[3]We can say that these integers are the topological degrees of the maps, if we fix the appropriate homotopy equivalences between $\mathbb{T}^1$, $\mathrm{SL}(2, \mathbb{R})$, and $\mathbb{P}^1$.

2.2. **Another example.** We will present an interesting example of a cocycle that has positive exponent without being uniformly hyperbolic. (Many more examples with those properties can be given using Theorem 3.4 below.)

Theorem 2.9 (Herman [12])**.** *Let T be a irrational rotation of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Fix $c > 1$ and let $A : X \to \mathrm{SL}(2, \mathbb{R})$ be given by*

$$A(x) = HR_{2\pi x} \quad \textit{where} \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad H = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix},$$

Then the following bound for the Lyapunov exponent holds:

$$\lambda \geq \log \frac{c + c^{-1}}{2}. \tag{2.4}$$

Note. In fact, it can be shown that equality holds in (2.4); see [3].

Thus if $c \neq 1$ the cocycle has positive Lyapunov exponent. Moreover, it is not uniformly hyperbolic, because (2.3) has no solution with $\mathbf{t} = 1$ and $\mathbf{a} = 1$.

The proof below uses subharmonic functions; see e.g. [17].

Proof. Let α so that the rotation T is $x \mapsto x + \alpha$. We have

$$\lambda = \lim_{n\to\infty} \frac{1}{2\pi n} \int_0^{2\pi} \log \|HR_{\theta+2\pi(n-1)\alpha} H \cdots HR_{\theta+2\pi\alpha} HR_\theta\|_* \, d\theta$$

Where $\|\cdot\|_*$ is the norm of the maximum absolute value of the entries. Define the following complex matrices:

$$T(z) = \begin{pmatrix} (z^2+1)/2 & (-z^2+1)/2i \\ (z^2-1)/2i & (z^2+1)/2 \end{pmatrix} \quad \text{for } z \in \mathbb{C}.$$

If $z = e^{i\theta}$ then $R_\theta = z^{-1}T(z)$. Write

$$C_n(z) = HT(e^{2\pi(n-1)\alpha i} z) \cdots HT(e^{2\pi\alpha i} z) HT(z).$$

Then

$$\lambda = \lim_{n\to\infty} \frac{1}{2\pi n} \int_0^{2\pi} \log \|C_n(e^{i\theta})\|_* \, d\theta.$$

The log of the absolute value of a holomorphic function is subharmonic, and the maximum of subharmonic functions is subharmonic; thus the function $z \in \mathbb{C} \mapsto \log \|C_n(z)\|_*$ is subharmonic. We obtain $\lambda \geq \lim \frac{1}{n} \log \|C_n(0)\|_*$. Now, $C_n(0) = [HT(0)]^n$ and the spectral radius of $HT(0)$ is $(c + c^{-1})/2$. So (2.4) follows. □

This kind of estimate is called *Herman's subharmonic trick*.

3. Products of i.i.d. matrices

3.1. **Statement of Furstenberg's Theorem.** We deal with Lyapunov exponents of products of random i.i.d. matrices. Let μ be a probability measure in $\mathrm{SL}(2,\mathbb{R})$ which satisfies the integrability condition

$$\int_{\mathrm{SL}(2,\mathbb{R})} \log \|M\| \, d\mu(M) < \infty.$$

Let $Y_1, Y_2, \ldots$ be random independent matrices with distribution μ, and let λ be the Lyapunov exponent, that is, the non-negative number so that

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \|Y_n \cdots Y_1\| \quad \text{almost surely.}$$

(The existence of the Lyapunov exponent follows from Theorem 1.3 applied to the cocycle described in Example 1.2; λ is constant a.e. because the base dynamics is ergodic.)

We will prove that $\lambda > 0$ for "most" choices of μ. Let us see some examples where $\lambda = 0$:

Example 3.1. If μ is supported in the group of rotations $\mathrm{SO}(2,\mathbb{R})$ then $\lambda = 0$.

Example 3.2. If μ is supported in the subgroup

$$\left\{ \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix};\ t, s \in \mathbb{R},\ t \neq 0 \right\}$$

then $\lambda = \left| \int \log \|M(1,0)\| \, d\mu(M) \right|$, which may be zero.

Example 3.3. Assume that only two matrices occur:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \text{and} \quad R_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then it is a simple exercise to show that $\lambda = 0$.

The result below says that the list above essentially covers all possibilities where the exponent vanishes:

Theorem 3.4 (Furstenberg [9])**.** *Assume μ is a probability measure on $\mathrm{SL}(2,\mathbb{R})$ satisfying the integrability condition and the following two assumptions:*

(i) *The support of μ is not contained in a set of the form $\{CRC^{-1};\ R \in \mathrm{SO}(2,\mathbb{R})\}$, where $C \in \mathrm{GL}(2,\mathbb{R})$.*
(ii) *There is no set $L \subset \mathbb{P}^1$ with 1 or 2 elements such that $M(L) = L$ for μ-a.e. M.*

Then the Lyapunov exponent λ is positive.

Note. The proof given here is based in [8]; also see that book for the higher-dimensional related results. More recently, this kind of results were extended to much broader classes of dynamical systems; see [4] and references therein.

The hypotheses of the theorem can be restated as follows:

(i′) The support of μ is not contained in a compact subgroup of $\mathrm{SL}(2,\mathbb{R})$;

(ii′) There is no finite set $\varnothing \neq L \subset \mathbb{P}^1$ such that $M(L) = L$ for μ-a.e. M.

Proof. Assumptions (i) and (i′) are equivalent. On the other hand, if $M \in \mathrm{SL}(2,\mathbb{R})$ fixes three different directions then $M = \pm Id$. Hence if $L \subset \mathbb{P}^1$ is a finite set with $\#L \geq 3$ then the set of $M \in \mathrm{SL}(2,\mathbb{R})$ such that $M(L) = L$ is finite. Therefore (i) and (ii) imply (ii′). □

3.2. **Proof.**

3.2.1. *Non-atomic measures in* $\mathbb{P}^1$. Let $\mathcal{M}(\mathbb{P}^1)$ be the space of probability Borel measures in $\mathbb{P}^1$. A measure $\nu \in \mathcal{M}(\mathbb{P}^1)$ is called *non-atomic* if $\nu(\{x\}) = 0$ for all $x \in \mathbb{P}^1$.

We collect some simple facts for later use.

If $A \in \mathrm{GL}(2,\mathbb{R})$ then we also denote by A the induced map $A\colon \mathbb{P}^1 \to \mathbb{P}^1$. If A is not invertible but $A \neq 0$ then there is only one direction $x \in \mathbb{P}^1$ for which Ax is not defined. In this case, it makes sense to consider the push-forward $A\nu \in \mathcal{M}(\mathbb{P}^1)$, if $\nu \in \mathcal{M}(\mathbb{P}^1)$ is non-atomic.

Lemma 3.5. *If $\nu \in \mathcal{M}(\mathbb{P}^1)$ is non-atomic and A_n is a sequence of non-zero matrices converging to $A \neq 0$, then $A_n\nu \to A\nu$ (weakly).*

The proof is easy.

Lemma 3.6. *If $\nu \in \mathcal{M}(\mathbb{P}^1)$ is non-atomic then*

$$H_\nu = \{M \in \mathrm{SL}(2,\mathbb{R});\ M\nu = \nu\}$$

is a compact subgroup of $\mathrm{SL}(2,\mathbb{R})$.

Proof. Assume that there exists a sequence M_n in H_ν with $\|M_n\| \to \infty$. Up to taking a subsequence, we may assume that the sequence $\|M_n\|^{-1}M_n$ converges to a matrix C. We have $\|C\| = 1$, so Lemma 3.5 gives $C\nu = \nu$. On the other hand,

$$\det C = \lim \frac{1}{\|M_n\|^2} = 0.$$

Thus C has rank one and $\nu = C\nu$ must be a Dirac measure, contradiction. □

3.2.2. *μ-invariant measures in $\mathbb{P}^1$.* If $\nu \in \mathcal{M}(\mathbb{P}^1)$, let the *convolution* $\mu * \nu \in \mathcal{M}(\mathbb{P}^1)$ is the push-forward of the measure $\mu \times \nu$ by the evaluation map $\mathrm{ev}\colon \mathrm{SL}(2,\mathbb{R}) \times \mathbb{P}^1 \to \mathbb{P}^1$. If $\mu * \nu = \nu$ then ν is called *μ-invariant*.

Example 3.7. Assume the probability μ is (left-)invariant by rotations, in the sense that $L_R\mu = mu$ for every $R \in \mathrm{SO}(2,\mathbb{R})$, where $L_R : M \mapsto RM$. Let ν_0 denote the Lebesgue measure on $\mathbb{P}^1$, that is, the normalized angle measure. Then for any $\nu \in \mathcal{M}(\mathbb{P}^1)$, we have $\mu * \nu = \nu_0$. (Indeed, for any $R \in \mathrm{SO}(2,\mathbb{R})$, we have $R(\mu * \nu) = (L_R\mu) * \nu = \mu * \nu$; so $\mu * \nu$ is invariant by rotations.) In particular, ν_0 is μ-invariant.

By a Krylov–Bogoliubov argument, μ-invariant measures always exist.

Lemma 3.8. *If μ satisfies the assumptions of Furstenberg's Theorem 3.4 then every μ-invariant $\nu \in \mathcal{M}(\mathbb{P}^1)$ is non-atomic.*

Proof. Assume that
$$\beta = \max_{x \in X} \mu(\{x\}) > 0.$$
Let $L = \{x;\ \mu(\{x\}) = \beta\}$. If $x_0 \in L$ then
$$\beta = \nu(\{x_0\}) = (\mu * \nu)(\{x_0\}) = \iint \chi_{\{x_0\}}(Mx)\, d\mu(M)\, d\nu(x)$$
$$= \int \nu(\{M^{-1}(x_0)\})\, d\mu(M).$$
But $\nu(\{M^{-1}(x_0)\}) \le \beta$ for all M, so $\nu(\{M^{-1}(x_0)\}) \le \beta$ for μ-a.e. M. We have proved that $M^{-1}(L) \subset L$ for μ-a.e. M. This contradicts assumption (ii′). ☐

From now on we assume that μ satisfies the assumptions of Furstenberg's Theorem 3.4, and that ν is a (non-atomic) μ-invariant measure in $\mathbb{P}^1$.

3.2.3. *The measure ν and the exponent λ.* The shift $\sigma\colon \mathrm{SL}(2,\mathbb{R})^{\mathbb{N}} \hookleftarrow$ in the space of sequences $\omega = (Y_1, Y_2, \ldots)$ has the ergodic invariant measure $\mu^{\mathbb{N}}$.

The skew-product map $T\colon \mathrm{SL}(2,\mathbb{R})^{\mathbb{N}} \times \mathbb{P}^1 \hookleftarrow$, $T(\omega, x) = (\sigma(\omega), Y_1(\omega)x)$ leaves invariant the measure $\mu \times \nu$. Consider $f\colon \mathrm{SL}(2,\mathbb{R})^{\mathbb{N}} \times \mathbb{P}^1 \to \mathbb{R}$ given by
$$f(\omega, x) = \log \frac{\|Y_1(\omega)x\|}{\|x\|}.$$
(The meaning is obvious). Then
$$\frac{1}{n}\sum_{j=0}^{n} f(T^j(\omega, x)) = \frac{1}{n} \log \frac{\|Y_n(\omega)\cdots Y_1(\omega)x\|}{\|x\|}.$$

By Oseledets' Theorem, for a.e. ω and for all $x \in \mathbb{P}^1$ except $x = E^-_\omega$ in the case $\lambda > 0$, the quantity on the right hand side tends to λ as $n \to \infty$. In particular, this convergence holds for $\mu^{\mathbb{N}} \times \nu$-a.e. (ω, x). We conclude that

$$\lambda = \iint f \, d\mu^{\mathbb{N}} \, d\nu = \iint \log \frac{\|Mx\|}{\|x\|} \, d\mu(M) \, d\nu(x). \tag{3.1}$$

Example 3.9. As in Example 3.7, assume that μ is (left-)invariant by rotations, and let $\nu = \nu_0$ be Lebesgue measure on $\mathbb{P}^1$. So (3.1) becomes:

$$\lambda = \frac{1}{2\pi} \int_{\mathrm{SL}(2,\mathbb{R})} \int_0^{2\pi} \log \|M(\cos\theta, \sin\theta)\| \, d\theta \, d\mu(M) \, .$$

The integral $\int_0^{2\pi} \log \sqrt{c^2 \cos^2\theta + c^{-2} \sin^2\theta} \, d\theta$ equals $2\pi \log \frac{c+c^{-1}}{2}$ (see [3]), therefore:

$$\lambda = \int_{\mathrm{SL}(2,\mathbb{R})} \log \frac{\|M\| + \|M\|^{-1}}{2} \, d\mu(M) \, .$$

In particular, the Lyapunov exponent is positive unless μ is concentrated on $\mathrm{SO}(2,\mathbb{R})$.

3.2.4. *Convergence of push-forward measures.* Let $S_n(\omega) = Y_1(\omega) \cdots Y_n(\omega)$. (Attention for the order of the product.)

Lemma 3.10. *For $\mu^{\mathbb{N}}$-a.e. ω, there exists $\nu_\omega \in \mathcal{M}(\mathbb{P}^1)$ such that*

$$S_n(\omega)\nu \to \nu_\omega.$$

For the proof we will need some background on probability (see e.g. [11]). Given a probability space $(X, \mathcal{F}, \mu)$, a *martingale* is composed of a sequence of random variables $X_1, X_2, \cdots \in L^1(\mu)$ and a sequence of σ-algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$ such that for each n, the function X_n is $\mathcal{F}_n$-measurable and $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n$ μ-almost surely. *Doob's Martingale Converge Theorem* asserts that there exists $X \in L^1(\mu)$ such that $X_n \to X$ μ-a.s.

Proof of Lemma 3.10. Fix $f \in C(\mathbb{P}^1, \mathbb{R})$. Associate to f the function $F \colon \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$ given by

$$F(M) = \int_{\mathbb{P}^1} f(Mx) \, d\nu(x).$$

Let $\mathcal{F}_n$ be the σ-algebra of $\mathrm{SL}(2,\mathbb{R})^{\mathbb{N}}$ formed by the cylinders of length n; then $S_n(\cdot)$ is $\mathcal{F}_n$-measurable. Also

$$\begin{aligned}
\mathbb{E}(F(S_{n+1}) \mid \mathcal{F}_n) &= \int F(S_n M)\, d\mu(M) \\
&= \iint f(S_n M x)\, d\mu(M)\, d\nu(x) \\
&= \int f(S_n y)\, d\nu(y) \qquad\qquad (\text{since } \mu * \nu = \mu) \\
&= F(S_n)\,.
\end{aligned}$$

This shows that the sequence of functions $\omega \mapsto F(S_n(\omega))$ is a martingale. Therefore the limit

$$\Gamma f(\omega) = \lim_{n\to\infty} F(S_n(\omega))$$

exists for a.e. ω.

Now let $\{f_k;\ k \in \mathbb{N}\}$ be a countable dense subset of $C(\mathbb{P}^1, \mathbb{R})$. Take ω in the full-measure set where $\Gamma f_k(\omega)$ exists for all k. Let ν_ω be a (weak) limit point of the sequence of measures $S_n(\omega)\nu$. Then

$$\int f_k\, d\nu_\omega = \lim_{i\to\infty} \int f_k\, d(S_{n_i}\nu) = \lim_{i\to\infty} \int f \circ S_{n_i}\, d\nu = \Gamma f_k(\omega).$$

Since the limit is the same for all subsequences, we have in fact that $S_n(\omega)\nu \to \nu_\omega$. □

Let's explore the construction of the measures to obtain more information about them:

Lemma 3.11. *For $\mu^{\mathbb{N}}$-a.e. ω, the measures ν_ω from Lemma 3.10 satisfy*

$$S_n(\omega) M \nu \to \nu_\omega \quad \textit{for } \mu\textit{-a.e. } M.$$

Proof. The proof is tricky. We have to show that, for any fixed $f \in C(\mathbb{P}^1, \mathbb{R})$, that[4]

$$\lim \mathbb{E}(F(S_n M)) = \Gamma f = \lim \mathbb{E}(F(S_n)) \quad \text{for } \mu\text{-a.e. } M \in \mathrm{SL}(2,\mathbb{R}). \tag{3.2}$$

We are going to show that

$$\lim_{n\to\infty} \mathbb{E}\left((F(S_{n+1}) - F(S_n))^2\right) = 0. \tag{3.3}$$

This is sufficient, because

$$\mathbb{E}\left((F(S_{n+1}) - F(S_n))^2\right) = \mathbb{E}\left(\left(\iint (f(S_n M x) - f(S_n x))\, d\nu(x)\, d\mu(M)\right)^2\right).$$

[4] $\mathbb{E}$ is integration on ω.

So (3.3) gives that, for a.e. ω,

$$\lim_{n\to\infty}\int (F(S_nM)-F(S_n))\,d\mu(M) = \lim_{n\to\infty}\iint (f(S_nMx)-f(S_nx))\,d\nu(x)\,d\mu(M) = 0.$$

This implies (3.2).

We have

$$\mathbb{E}\left((F(S_{n+1})-F(S_n))^2\right) = \mathbb{E}(F(S_{n+1})^2) + \mathbb{E}(F(S_n)^2) - 2\mathbb{E}(F(S_{n+1})F(S_n)).$$

But

$$\mathbb{E}(F(S_{n+1})F(S_n)) = \mathbb{E}\left(\int f\circ S_{n+1}\,d\nu\cdot\int f\circ S_n\,d\nu\right) =$$
$$\mathbb{E}\left(\iint f(S_nMx)\,d\nu(x)\,d\mu(M)\cdot\int f\circ S_n\,d\nu\right) =$$
$$\mathbb{E}\left(\left(\int f\circ S_n\,d\nu\right)^2\right) = \mathbb{E}(F(S_n)^2).$$

So

$$\mathbb{E}\left((F(S_{n+1})-F(S_n))^2\right) = \mathbb{E}(F(S_{n+1})^2) - \mathbb{E}(F(S_n)^2).$$

Hence, by cancelation, for any p,

$$\sum_{n=1}^{p}\mathbb{E}\left((F(S_{n+1})-F(S_n))^2\right) = \mathbb{E}(F(S_{p+1})^2) - \mathbb{E}(F(S_1)^2) \le \|f\|_\infty^2.$$

Therefore $\sum_{n=1}^{p}\mathbb{E}\left((F(S_{n+1})-F(S_n))^2\right) < \infty$ and (3.3) follows. □

3.2.5. *The limit measures are Dirac.*

Lemma 3.12. *For $\mu^{\mathbb{N}}$-a.e. ω, there exists $Z(\omega)\in\mathbb{P}^1$ such that $\nu_\omega = \delta_{Z(\omega)}$.*

Proof. Fix ω. We have, for μ-a.e. M,

$$\lim S_n\nu = \lim S_nM\nu.$$

Let B be a limit point of the sequence of norm 1 matrices $\|S_n\|^{-1}S_n$. Since $\|B\|=1$, we can apply Lemma 3.5:

$$B\nu = BM\nu.$$

If B were invertible, this would imply $\nu = M\nu$. That is, a.e. M belongs to the compact group H_ν (see Lemma 3.6), contradicting hyphotesis (i). So B is non-invertible. Since $B\nu = \nu_\omega$, we conclude that ν_ω is Dirac. □

3.2.6. *Convergence to Dirac implies norm growth.*

Lemma 3.13. *Let $m \in \mathcal{M}(\mathbb{P}^1)$ be non-atomic and let (A_n) be a sequence in* $\mathrm{SL}(2, \mathbb{R})$ *such that $A_n m \to \delta_z$, where $z \in \mathbb{P}^1$. Then*

$$\|A_n\| \to \infty.$$

Moreover, for all $v \in \mathbb{R}^2$,

$$\frac{\|A_n^*(v)\|}{\|A_n\|} \to |\langle v, z\rangle|.$$

Proof. If $\|A_n\|$ does not converge to ∞ then there is a converging subsequence $A_{n_i} \to \tilde{A}$. But then $\tilde{A}m = \delta_z$ would not be atomic.

Write $A_n = R_{\alpha_n} H_{c_n} R_{\beta_n}$, where $c_n = \|A_n\|$ and H_c is the diagonal matrix with c, c^{-1} along the diagonal. We claim that $\alpha_n \to z$. If this were not the case, we could pass to a subsequence so that $\alpha_n \to \alpha \neq z$ and $\beta_n \to \beta$. Then $B_n = A_n/\|A_n\|$ converges to the non-invertible B whose range is α. By Lemma 3.5 we have $B_n m \to Bm = \delta_\alpha$, so $\alpha = z$.

Now the claim follows straightforwardly. □

3.2.7. *Norm growth cannot be slower than exponential.* We shall use the following abstract lemma from ergodic theory:

Lemma 3.14. *Let $T\colon (X, m) \hookleftarrow$ be a measure preserving transformation of a probability space (X, m). If $f \in L^1(m)$ is such that*

$$\sum_{j=0}^{n-1} f(T^j x) = +\infty \quad \textit{for } m\textit{-almost every } x,$$

then $\int f \, dm > 0$.

Proof. Let $\tilde{}$ denote limit of Birkhoff averages. Then $\tilde{f} \geq 0$. Assume, by contradiction, that $\int f = 0$. Then $\tilde{f} = 0$ a.e.

Let $s_n = \sum_{j=0}^{n-1} f \circ T^j$. For $\varepsilon > 0$, let

$$A_\varepsilon = \{x \in X;\ s_n(x) \geq \varepsilon \ \forall n \geq 1\} \quad \text{and} \quad B_\varepsilon = \bigcup_{k \geq 0} T^{-k}(A_\varepsilon).$$

Fix $\varepsilon > 0$ and let $x \in B_\varepsilon$. Let $k = k(x) \geq 0$ be the least integer such that $T^k x \in A_\varepsilon$. We compare the Birkhoff sums of f and χ_{A_ε}:

$$\sum_{j=0}^{n-1} f(T^j x) \geq \sum_{j=0}^{k-1} f(T^j x) + \sum_{j=k}^{n-1} \varepsilon \chi_{A_\varepsilon}(T^j x) \quad \forall n \geq 1.$$

Dividing by n and making $n \to \infty$ we get

$$0 = \tilde{f}(x) \geq \varepsilon \widetilde{\chi_{A_\varepsilon}}(x)$$

Therefore

$$\mu(A_\varepsilon) = \int \widetilde{\chi_{A_\varepsilon}} = \int_{B_\varepsilon} \widetilde{\chi_{A_\varepsilon}} = 0.$$

Thus $\mu(B_\varepsilon) = 0$ for every $\varepsilon > 0$ as well.

On the other hand, if $s_n(x) \to \infty$ then $x \in \bigcup_{\varepsilon>0} B_\varepsilon$. We have obtained a contradiction. □

End of the proof of Theorem 3.4. Replace everywhere Y_i by Y_i^*. Note that μ^* also satisfies the hypothesis of the theorem if μ does.[5]

Let T and f be as in §3.2.3. By Lemmas 3.12 and 3.13 we have

$$\sum_{j=0}^{n} f(T^j(\omega, x)) = \log \frac{\|S_n^*(\omega)x\|}{\|x\|} \to \infty$$

for a.e. ω and all $x \in \mathbb{P}^1 \smallsetminus \{Z(\omega)^\perp\}$. In particular, convergence holds $\mu^{\mathbb{N}} \times \nu$-a.e. By Lemma 3.14, this implies $\int f > 0$. Then, by (3.1), $\lambda > 0$. □

4. Zero Lyapunov Exponents

The Furstenberg Theorem 3.4 leads to the impression that very few cocycles with zero Lyapunov exponents exist. The following is a result that goes in the opposite direction:

Theorem 4.1. *Let T be a homeomorphism of a compact Hausdorff space X, and let μ be a T-invariant ergodic fully supported probability measure. Then there is a residual subset $\mathcal{R}$ of the space $C(X, \mathrm{SL}(2,\mathbb{R}))$ such that for any $A \in \mathcal{R}$, either (T, A) is uniformly hyperbolic or the Lyapunov exponent of the cocycle (T, A) which respect to the measure μ is zero.*

Here $C(X, \mathrm{SL}(2,\mathbb{R}))$ is the space of continuous maps $X \to \mathrm{SL}(2,\mathbb{R})$ endowed with the topology of uniform convergence. It is a Baire space, therefore every residual[6] set is dense.

Note. Theorem 4.1 was proved in [6], which also contains its (harder) version for area-preserving diffeomorphisms. The proof presented here is specific for cocycles and follows [5]. For results for higher dimensional cocycles and diffeomorphisms, see [7].

Recall from Example 2.8 that there may exist non-empty open sets of $C(X, \mathrm{SL}(2,\mathbb{R}))$ formed by A's such that (T, A) is not uniformly hyperbolic. Thus Theorem 4.1 gives many cocycles with zero exponents. For example, the cocycle from Theorem 2.9 can be perturbed (keeping the base dynamics unchanged) so that the exponent becomes zero.

[5]Because $A(v) = w \Rightarrow A^*(w^\perp) = v^\perp$.

[6]A residual set is a countable intersection of open and dense sets.

Example 4.2. It is necessary to assume that μ is fully supported in Theorem 4.1. (Notice the definition of uniform hyperbolicity in [6] is different and automatically entails this.) Let T be a homeomorphism of the circle homotopic to the identity and with an unique fixed point$p$. Let $A : \mathbb{T}^1 \to \mathrm{SL}(2, \mathbb{R})$ be not homotopic to constant and such that $A(p)$ is a hyperbolic matrix. (These conditions define an open subset of $C(X, \mathrm{SL}(2, \mathbb{R}))$.) In the notation of Example 2.8, we have $\mathbf{t} = 1$, $\mathbf{a} \neq 0$, thus (2.3) has no solution and (T, A) is not uniformly hyperbolic. The unique T-invariant probability measure μ is supported on the point p, so the associated Lyapunov exponent is positive.

4.1. **Proof.** In all this section we assume X, T, and μ are as in the hypotheses of Theorem 4.1.

4.1.1. *Semicontinuity.*

Lemma 4.3. *The Lyapunov exponent is an upper-semicontinuous function*

$$\lambda : C(X, \mathrm{SL}(2, \mathbb{R})) \to [0, \infty).$$

Proof. This is an easy consequence of (1.4) in Theorem 1.3. □

Another semicontinuity property that we will use is:

Lemma 4.4. *Given $A \in L^\infty(X, \mathrm{SL}(2, \mathbb{R}))$, $M > \|A\|_\infty$, and $\delta > 0$ there exists $\eta > 0$ such that*

$$\|B\|_\infty < M, \quad \mu\{x \in X;\ B(x) \neq A(x)\} < \eta \quad \Rightarrow \quad \lambda(B) < \lambda(A) + \delta.$$

Proof. Let A, M, and δ be given. Take n such that

$$\frac{1}{n} \int_X \log \|A^n(x)\| \, d\mu(x) < \lambda(A) + \frac{\delta}{2} .$$

Let η be very small, and let B be such that $\|B\|_\infty < M$ and the set $R = [B \neq A]$ has measure less than η. Then $S = \bigcup_{j=0}^{n-1} T^{-j} R$ has measure $< n\eta$. We estimate

$$\begin{aligned} \lambda(B) \leq \frac{1}{n} \int \log \|B^n\| = \frac{1}{n} \int_S \log \|B^n\| + \frac{1}{n} \int_{X \setminus S} \log \|A^n\| \\ \leq (\log M)\mu(S) + \lambda(A) + \frac{\delta}{2} < \lambda(A) + \delta \, , \end{aligned}$$

provided η was chosen small enough. □

Note. For semicontinuity in a stronger sense (that is, with respect to a weaker topology), see [1].

4.1.2. *Interchanging the Oseledets Directions.*

Lemma 4.5. *If T is an aperiodic invertible transformation, Y is a measurable set with $\mu(Y) > 0$ and $n \geq 1$, then there exists $Z \subset Y$ with $\mu(Z) > 0$ and such that $Z, TZ, \ldots, T^n Z$ are disjoint sets.*

The set $Z = \bigsqcup_{j=0}^{n} T^j Z$ is called a *tower of height* $n + 1$.

Proof. Take $Y_1 \subset Y$ such that $\mu(Y_1 \vartriangle TY_1) > 0$ (it exists because otherwise a.e. point of Y would be fixed). Then $Z_1 = Y_1 \smallsetminus TY_1$ has positive measure and $Z_1 \cap T(Z_1) = \varnothing$. Take $Y_2 \subset Z_1$ such that $\mu(Y_2 \vartriangle T^2(Y_2)) > 0$ and let $Z_2 = Y_2 \smallsetminus f^2(Y_2)$. Continuing in this way we will find $Z = Z_n$ such that Z, TZ, $\ldots$, $T^n Z$ are two-by-two disjoint. ◻

Lemma 4.6. *Assume $A \in C(X, \mathrm{SL}(2, \mathbb{R}))$ is such that the cocycle (T, A) has positive exponent and is not uniformly hyperbolic. Then for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$, a measurable set $Y \subset X$ of positive measure such that Y, $TY, \ldots, T^{m-1}Y$ are disjoint, and a measurable map $B : X \to \mathrm{SL}(2, \mathbb{R})$ with the following properties:*

- $\|B(x) - A(x)\| < \varepsilon$ *everywhere;*
- *B equals A outside of $\bigsqcup_{j=0}^{m-1} T^j Y$;*
- *$B^m(x) \cdot E^+_x = E^-_{T^m x}$ and $B^m(x) \cdot E^-_x = E^+_{T^m x}$ for all $x \in Y$, where E^+, E^- denote the Oseledets directions of the unperturbed cocyle (T, A).*

Proof. The proof has two parts: In the first part, we will see how to send E^+ to E^-, without caring about the image of E^-. More precisely, we will find m, Y, and B satisfying all the desired conclusions except for $B^m(x) \cdot E^-_x = E^+_{T^m x}$. Then in the second part of the proof, we will see how to modify the construction and obtain $\tilde{m}$, $\tilde{Y}$, and $\tilde{B}$ with all the desired properties.

First part: sending E^+ to E^-. Fix ε' with $0 < \varepsilon' \ll \varepsilon$. If the set of points x where $\measuredangle(E^+(x), E^-(x)) < \varepsilon'$ has positive measure then we are done: take Y as that set, $m = 1$, and for $x \in Y$, let $B(x)$ be $A(x)$ composed with a small rotation in order to have $B(x) \cdot E^+_x = E^-_{Tx}$.

So assume the angle between E^+ and E^- is at least ε' almost everywhere.

For $m \in \mathbb{N}$, consider the set

$$\Gamma_m = \left\{ x \in X;\ \frac{\|A^m(x)|E^-_x\|}{\|A^m(x)|E^+_x\|} > \frac{1}{2} \right\} .$$

We claim that $\mu(\Gamma_m) > 0$ for every m. To show this, assume on the contrary that there is m such that

$$\frac{\|A^m(x)|E_x^-\|}{\|A^m(x)|E_x^+\|} \leq \frac{1}{2} \quad \text{for a.e. } x \in X.$$

Then for any $k \geq 1$ and a.e. $x \in X$,

$$\|A^{km}(x)\|^2 \geq \frac{\|A^{km}(x)|E_x^+\|}{\|A^{km}(x)|E_x^-\|} \geq 2^k\,.$$

By continuity, $\|A^{km}(x)\| \geq 2^{k/2}$ for all x in the support of the measure μ, which by assumption is the whole space X. This implies that (T, A) is uniformly hyperbolic, contrary to the assumption.

Now fix $m \gg 1$. Use Lemma 4.5 and take $Y \subset \Gamma_m$ with positive measure such that $Y, TY, \ldots, T^{m-1}Y$ are disjoint. For each $x \in Y$ and j with $0 < j < m-1$, let $B(T^j x)$ be $A(T^j x)$ composed with a linear map close to the identity that fixes the directions $E^+_{T^j x}$ and $E^-_{T^j x}$, expanding the former and contracting the latter by some definite factor. (Here we use that there is a lower bound on $\measuredangle(E^+, E^-)$.) This gives

$$\frac{\|A(T^{m-1}x)B(T^{m-2}x)\cdots B(Tx)A(x)|E_x^-\|}{\|A(T^{m-1}x)B(T^{m-2}x)\cdots B(Tx)A(x)|E_x^+\|} \gg 1.$$

This implies that there is a direction close to E^+_{Tx} which is sent by $B(T^{m-2}x)\cdots B(Tx)$ to a direction close to $E^-_{T^{m-1}x}$. Now define $B(x)$ and $B(T^{m-1}x)$ respectively as $A(x)$ and $A(T^{m-1}x)$ composed with appropriate small rotations in order to have $B^m(x)\cdot E_x^+ = E^-_{T^m x}$.

Second part: sending E^- to E^+. We had found B, m, and Y such that for $x \in Y$, $B^m(x)\cdot E_x^+ = E^-_{T^m x}$ and, in particular, $B^m(x)\cdot E_x^- \neq E^-_{T^m x}$. Hence (recall (1.12))

$$\measuredangle\Big(A^n(T^m x)\cdot B^m(x)\cdot E_x^-, E^+_{T^{m+n}x}\Big) \to 0 \quad \text{as } n \to \infty.$$

On the other hand, by Poincaré recurrence,

$$\limsup_{n\to\infty} \measuredangle\Big(E^-_{T^{m+n}x}, E^+_{T^{m+n}x}\Big) > 0.$$

Thus we can find some n and some positive measure set $\tilde{Y} \subset Y$ such that for all $x \in \tilde{Y}$, $\measuredangle\Big(A^n(T^m x)\cdot B^m(x)\cdot E_x^-, E^+_{T^{m+n}x}\Big)$ is very small, while $\Big(E^-_{T^{m+n}x}, E^+_{T^{m+n}x}\Big)$ is not very small. By Lemma 4.5, we can assume $\tilde{Y} \cap T^j\tilde{Y} = \emptyset$ for all j with $1 \leq j \leq n+m$. Let us define a map $\tilde{B} : X \to \mathrm{SL}(2,\mathbb{R})$ as follows: On $\bigsqcup_{j=0}^{m-1} T^j\tilde{Y}$, $\tilde{B}$ equals B. For each $x \in Y$, let $\tilde{B}(T^{m+n-1}x)$ be $A(T^{m+n-1}x)$ composed with a linear map close to the identity that fixes $E^-_{T^{m+n}x}$ and sends $A^n(T^m x)\cdot B^m(x)\cdot E_x^-$ to $E^+_{T^{m+n}x}$.

Finally, over the rest of X, let $\tilde{B}$ equal A. Then $\tilde{m} = m + n + 1$, $\tilde{Y}$, and $\tilde{B}$ have all the desired properties. This proves Lemma 4.6. □

4.1.3. *Vanishing of the Exponent.* We write $Z \doteq Z'$ if Z and Z' are measurable sets and their symmetric difference $Z \vartriangle Z'$ has zero measure. A measurable set $Z \subset X$ is called a *coboundary* if there exists a measurable set $W \subset X$ such that $Z \doteq W \vartriangle TW$.

Lemma 4.7. *Assume T is an ergodic automorphism of the Lebesgue space X. Given a set $Y \subset X$ with positive measure, there exists a set $Z \subset Y$ which is not a coboundary.*

Proof. See [15]. □

The following key lemma is essentially due to Knill [15]:

Lemma 4.8. *If B is the map given by Lemma 4.6 and the set Y is not a coboundary then the Lyapunov exponent of (T, B) is zero.*

(Compare with Example 3.3.)

Proof. We analyze two cases separately.

First case: $m = 1$

First we need to make some general considerations. One can define the skew-product

$$\Phi_A : (x, \overline{v}) \in X \times \mathbb{P}^1 \mapsto (T(x), \overline{A(x) \cdot v}) \in X \times \mathbb{P}^1 .$$

If $\lambda(A) > 0$ then there are two measures μ^+ and μ^- that are invariant for Φ_A, given by

$$\mu^{\pm}(B) = \mu \{x \in X; (x, E^{\pm}_x) \in B\}$$

If $\pi : X \times \mathbb{P}^1 \to X$ denotes the projection on the first coordinate then $\pi_*(\mu^+) = \pi_*(\mu^-) = \mu$ and we say that μ^+ and μ^- project on μ.

Claim. If μ is ergodic and $\lambda(A) > 0$ then there are only two ergodic measures for Φ_A which project on μ, namely μ^+ and μ^-.

Proof. Let η be an ergodic measure for Φ_A which projects on μ. Let us define a function $f : X \times \mathbb{P}^1 \to \mathbb{R}$ by

$$f(x, \overline{v}) = \log \frac{\|A(x) \cdot v\|}{\|v\|}$$

For μ-a.e. $x \in X$ and all $v \in \mathbb{R}^2 \smallsetminus \{0\}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ \Phi_A^j(x, \overline{v}) = \begin{cases} \lambda(A) & \text{if } v \notin E^-(x) \\ -\lambda(A) & \text{if } v \in E^-(x) \end{cases}$$

Therefore, by Birkhoff's theorem,

$$\eta\left\{(x,\overline{v}) \in X \times \mathbb{P}^1; v \notin E^-(x)\right\} = 0 \text{ or } 1.$$

By the same reasoning,

$$\eta\left\{(x,\overline{v}) \in X \times \mathbb{P}^1; v \notin E^+(x)\right\} = 0 \text{ or } 1.$$

Thus the only possibilities are $\eta = \mu^+$ or $\eta = \mu^-$ and the claim is proved. □

We now return to the proof of the lemma. The skew-product $T \times B$ has the invariant measure

$$\hat{\mu} = \frac{1}{2}(\mu^+ + \mu^-).$$

Claim. $\hat{\mu}$ is an ergodic measure for Φ_B.

Proof. Assume that there exists a measurable set $Q \subset X \times \mathbb{P}^1$ with $0 < \hat{\mu}(Q) < 1$ which is invariant for Φ_B. For each $x \in X$, denote $Q_x = \{\overline{v} \in \mathbb{P}^1; (x,\overline{v}) \in Q\}$. By definition of $\hat{\mu}$ we can suppose that $Q_x \subset \{E^+(x), E^-(x)\}$ for every x. Since $\pi(Q)$ is T-invariant, we have $Q_x \neq \emptyset$. Further, the T-invariant set $\{x; Q_x = \{E^+(x), E^-(x)\}\}$ must have μ-measure zero. To simplify notations, let us write $Q_x = +$ or $-$, with the obvious meanings. Let

$$W = \{x \in X; Q_x = +\}.$$

Then

$$W \vartriangle T^{-1}W = \{x \in X;\ Q_x = +,\ Q_{Tx} = -\} \cup \{x \in X;\ Q_x = -,\ Q_{Tx} = +\} = Z.$$

This contradicts the assumption that Z is not a coboundary. □

Finally, if we had $\lambda(B) > 0$ then $\hat{\mu}$ would be a measure of the type given by the first claim. This is clearly impossible.

Second case: $n > 1$

We will reduce this case to the first one, but again some background information is needed. Given a measurable set, one can define the induced first-return system (U, T_U, μ_U) as follows:

$$x \in U \Rightarrow r(x) = \min\{n \geq 1; T^n x \in U\} \text{ and } T_U(x) = T^{r(x)}x,$$

$$V \subset U \Rightarrow \mu_U(V) = \frac{\mu(V)}{\mu(U)}.$$

We can also define a induced cocycle A_U over this system as

$$x \in U \Rightarrow A_U(x) = A^{r(x)}(x).$$

By [15, Lemma 2.2], the system (U, T_U, μ_U) is ergodic and

$$\lambda(A_U) = \frac{\lambda(A_U)}{\mu(U)}.$$

Let $U = X \smallsetminus (TZ \cup \cdots \cup T^{n-1}Z) \supset Z$. It follows from [15, Lemma 3.4] that Z is not a coboundary for (U, T_U, μ_U).

We have

$$x \in U \;\Rightarrow\; B_U(x) = \begin{cases} B^n(x) & \text{if } x \in Z \\ A(x) & \text{otherwise} \end{cases}$$

Hence the first case guarantees that $\lambda(B_U) = 0$ and therefore $\lambda(B) = 0$. □

4.1.4. *Conclusion of the proof.*

Proof of Theorem 4.1. Let $\mathcal{H}$, resp. $\mathcal{U}_\delta$, is the set of A's such that (T, A) is uniformly hyperbolic, resp. the Lyapunov exponent is less than δ. Then $\mathcal{H}$ and $\mathcal{U}_\delta$ are open subsets of $C(X, \mathrm{SL}(2,\mathbb{R}))$, respectively by Proposition 2.6 and Lemma 4.3. To prove the Theorem, we will show that for every $\delta > 0$, $\mathcal{H} \cup \mathcal{U}_\delta$ is dense in $C(X, \mathrm{SL}(2,\mathbb{R}))$. It is sufficient to show that given $A \notin \mathcal{H}$ and $\delta > 0$, there exists $\tilde{A} \in \mathcal{U}_\delta$ arbitrarily close to A.

So fix A and δ as above, and let $\varepsilon > 0$. We can assume T is aperiodic. Let the cocycle B and the set Y be given by Lemma 4.6, with $\|B - A\|_\infty < \varepsilon$. Because of Lemma 4.7, we can assume Y is not a coboundary. Thus Lemma 4.8 gives that $\lambda(B) = 0$. Take $M \gg \|B\|_\infty$. Using Lusin's theorem (see [17], for instance), we see that can find a continuous $\tilde{A} : X \to \mathrm{SL}(2,\mathbb{R})$ that differs from B only in a set of very small measure, and such that $\|\tilde{A} - A\|_\infty = O(\varepsilon)$. By Lemma 4.4 it follows that $\lambda(\tilde{A}) < \delta$. □

References

[1] A. Arbieto and J. Bochi. L^p-generic cocycles have one-point Lyapunov spectrum. *Stoch. Dyn.*, 3(1):73–81, 2003.
[2] L. Arnold. *Random dynamical systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
[3] A. Avila and J. Bochi. A formula with some applications to the theory of Lyapunov exponents. *Israel J. Math.*, 131:125–137, 2002.
[4] A. Avila and M. Viana. Simplicity of Lyapunov spectra: a sufficient criterion. *Port. Math. (N.S.)*, 64(3):311–376, 2007.
[5] J. Bochi. Discontinuity of the Lyapunov exponent for non-hyperbolic cocycles. Unpublished, 1999.
[6] J. Bochi. Genericity of zero Lyapunov exponents. *Ergodic Theory Dynam. Systems*, 22(6):1667–1696, 2002.

[7] J. Bochi and M. Viana. The Lyapunov exponents of generic volume-preserving and symplectic maps. *Ann. of Math. (2)*, 161(3):1423–1485, 2005.

[8] P. Bougerol and J. Lacroix. *Products of random matrices with applications to Schrödinger operators*, volume 8 of *Progress in Probability and Statistics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[9] H. Furstenberg. Noncommuting random products. *Trans. Amer. Math. Soc.*, 108:377–428, 1963.

[10] H. Furstenberg and H. Kesten. Products of random matrices. *Ann. Math. Statist*, 31:457–469, 1960.

[11] G. R. Grimmett and D. R. Stirzaker. *Probability and random processes*. Oxford University Press, New York, third edition, 2001.

[12] M.-R. Herman. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnol'd et de Moser sur le tore de dimension 2. *Comment. Math. Helv.*, 58(3):453–502, 1983.

[13] Y. Katznelson and B. Weiss. A simple proof of some ergodic theorems. *Israel J. Math.*, 42(4):291–296, 1982.

[14] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B*, 30:499–510, 1968.

[15] O. Knill. The upper Lyapunov exponent of $SL(2, \mathbf{R})$ cocycles: discontinuity and the problem of positivity. In *Lyapunov exponents (Oberwolfach, 1990)*, volume 1486 of *Lecture Notes in Math.*, pages 86–97. Springer, Berlin, 1991.

[16] V. I. Oseledec. A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems. *Trudy Moskov. Mat. Obšč.*, 19:179–210, 1968.

[17] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[18] P. Thieullen. Ergodic reduction of random products of two-by-two matrices. *J. Anal. Math.*, 73:19–64, 1997.

[19] J.-C. Yoccoz. Some questions and remarks about $SL(2, \mathbf{R})$ cocycles. In *Modern dynamical systems and applications*, pages 447–458. Cambridge Univ. Press, Cambridge, 2004.

[20] L.-S. Young. Ergodic theory of differentiable dynamical systems. In *Real and complex dynamical systems (Hillerød, 1993)*, volume 464 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 293–336. Kluwer Acad. Publ., Dordrecht, 1995.

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