ERGODIC MEASURES WITH INFINITE ENTROPY

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ABSTRACT. We construct ergodic probability measures with infinite metric entropy for typical continuous maps and homeomorphisms on compact manifolds. We also construct sequences of such measures that converge to a zero-entropy measure.

1. Introduction

Let M be a compact manifold with finite dimension $m \geq 1$, with or without boundary. Let $C^0(M)$ be the space of continuous maps $f: M \to M$ with the uniform norm:

$$\|f - g\|_{C^0} := \max_{x \in M} \ \text{dist(f(x), g(x))} \quad \forall\ f, g \in C^0(M).$$

We denote by $\text{Hom}(M)$ the space of homeomorphisms $f: M \to M$ with the uniform norm:

$$\|f - g\|_{Hom} := \max\Big\{\|f - g\|_{C^0},\ \|f^{-1} - g^{-1}\|_{C^0}\Big\} \quad \forall\ f, g \in \text{Hom}(M).$$

A subset $\mathcal{S} \subset C^0(M)$ (or $\mathcal{S} \subset \text{Hom}(M)$) is called *a G_δ-set* if it is the countable intersection of open subsets of $C^0(M)$ (resp. $\text{Hom}(M)$). We say that a property P of the maps $f \in C^0(M)$ (or $f \in \text{Hom}(M)$) is *typical,* or that typical maps satisfy P, if the set of maps that satisfy P contains a *dense G_δ-set* in $C^0(M)$ (resp. $\text{Hom}(M)$).

The main result of this article is the following theorem.

Theorem 1. *The typical map $f \in C^0(M)$ has an ergodic Borel probability measure μ such that $h_\mu(f) = +\infty$.*

In the case that M is a compact interval, Theorem 1 was proved in [CT]. Yano proved that typical continuous maps of compact manifolds with or without boundary have infinite topological entropy [Ya]. Therefore, from the variational principle, there exists invariant measures with metric entropies as large as wanted. Nevertheless, this property alone

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does not imply the existence of invariant measures with infinite metric entropy. In fact, it is well known that the metric entropy function $\mu \to h_\mu(f)$ is not upper semi-continuous for C^0-typical systems. Moreover, we prove that it is *strongly* non upper semi-continuous in the following sense:

Theorem 2. *For a typical map $f \in C^0(M)$ there exists a sequence of ergodic measures μ_n such that*

$$h_{\mu_n} = +\infty \ \ \forall\, n \geq 1, \qquad \lim_{n\to+\infty}{}^* \mu_n = \mu, \qquad h_\mu = 0,$$

where $\lim^*$ *denotes the limit in the space of probability measures endowed with the weak* topology.*

Even if it were a-priori known that some f-invariant measure μ has infinite metric entropy, this property alone would not imply the existence of ergodic measures with infinite metric entropy as Theorems 1 and 2 state. Actually, if μ had infinitely many ergodic components, to prove that the metric entropy of at least one of its ergodic components must be larger or equal than the entropy of μ, one needs again the upper semi-continuity of the metric entropy function (see for instance [Ke, Theorem 4.3.7, p. 75]).

Yano also proved that typical homeomphisms on manifolds of dimension 2 or larger, have infinite topological entropy [Ya]. Thus one wonders if Theorems 1 and 2 hold also for homeomorphisms. We give a positive answer to this question.

Theorem 3. *If the dimension of the manifold M is at least 2, then the typical homeormorphism $f \in$ Hom(M) has an ergodic Borel probability measure μ such that $h_\mu(f) = +\infty$.*

Theorem 4. *If the dimension of the manifold M is at least 2, then for a typical homeomorphism $f \in$ Hom(M) there exists a sequence of ergodic measures μ_n such that*

$$h_{\mu_n} = +\infty \ \ \forall\, n \geq 1, \qquad \lim_{n\to+\infty}{}^* \mu_n = \mu, \qquad h_\mu = 0.$$

1.1. **Open questions.** If f is Lipschiz then no invariant measure has infinite entropy, since its topological entropy is finite. The following question arises: do Theorems 1 and 3 hold also for maps with more regularity than continuity but lower regularity than Lipschiz? For instance, do they hold for Hölder-continuous maps?

"A-priori" there is a chance to answer positively this question for one-dimensional Hölder continuous endomorphisms, because in such a case, the topological entropy is typically infinite [Ha]. Also for bi-Hölder homeomorphisms on manifolds of dimension 2 or larger, there is a chance to answer positively the above question, because their topological entropy is also typically infinite [FHT], [FHT1]. In this article we will focus only on the C^0-case, and leave for further research the eventual adaptation of our proofs, if this adaptation is possible, to C^α-maps for homeomorphisms with $0 < \alpha < 1$.

The hypothesis of Theorems 1 and 3 states that M is a compact manifold. It arises the following question: do some of the results also hold in other compact metric spaces that are not manifolds? For instance, do they hold if the space is a Cantor set K?

If the aim were just to construct $f \in Hom(K)$ with ergodic measures with infinite metric entropy, the answer would be positive. But if the purpose were to prove that such homeomorphisms are typical in $Hom(K)$, the answer would be negative.

In fact, Theorem 3 holds in particular for the 2-dimensional square $D^2 := [0, 1]^2$. One of the steps of the proofs consists in constructing some fixed Cantor set $\Lambda \subset D^2$, and a homeomorphism h on M that leaves Λ invariant, and possesses an h-invariant ergodic measure supported on Λ with infinite metric entropy (see Lemma 3.1 and Remark 3.2). Since any pair of Cantor sets K and Λ are homeomorphic, we deduce that any Cantor set K supports a homeomorphism f and an f-ergodic measure with infinite metric entropy.

Nevertheless, the above phenomenon is not typical on a Cantor set K. On the one hand, there also exists homeomorphisms on K with finite, and even zero, topological entropy. (Take for instance $f \in Hom(K)$ conjugated to the homeomorphism on the attractor of a Smale horseshoe, or to the attractor of the C^1- Denjoy example on the circle.) On the other hand, it is known that each homeomorphism on a Cantor set K is topologically locally unique; i.e., it is conjugated to any of its small perturbations [AGW]. Therefore, the topological entropy is locally constant in $Hom(K)$. We conclude that the homeomorphisms on the Cantor set K with infinite metric entropy, that do exist, are not dense in $Hom(K)$; hence they are not typical.

1.2. **Organisation of the article.** We construct a family $\mathcal{H}$, called models, of continuous maps in the cube $[0, 1]^m$, including some homeomorphisms of the cube onto itself if $m \geq 2$, which have complicated behavior on a Cantor set (Definition 2.6). A periodic shrinking box is a compact set $K \subset M$ that is homeomorphic to the cube $[0, 1]^m$ and such that for some $p \geq 1$: $K, f(K), \dots, f^{p-1}(K)$ are pairwise disjoint and $f^p(K) \subset \text{int}(K)$ (Definition 4.1). The main steps of the proofs of Theorems 1 and 3 are the following results.

Lemma 3.1 *Any model $h \in \mathcal{H}$ in the cube $[0,1]^m$ has an h-ergodic measure ν such that $h_\nu(h) = +\infty$*

Lemmas 4.2 and 4.5 *Typical maps in $C^0(M)$, and typical homeormorphisms, have a periodic shrinking box.*

Lemmas 4.8 and 4.9 *Typical maps $f \in C^0(M)$, and typical homeomorphisms for $m \geq 2$, have a periodic shrinking box K such that the return map $f^p|_K$ is topologically conjugated to a model $h \in \mathcal{H}$.*

A good sequence of periodic shrinking boxes is a sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes such that accumulate (with the Hausdorff distance) on a periodic point x_0, and besides their iterates $f^j(K_n)$ also accumulate on the periodic orbit of x_0, uniformly for $j \geq 0$ (see Definition 5.1). The main step in the proof of Theorems 2 and 4 is Lemma 3.1 together with

Lemma 5.2 *Typical maps $f \in C^0(M)$, and if $m \geq 2$ also typical homeomorphisms, have a good sequence $\{K_n\}$ of boxes, such that the return maps $f^{p_n}|_{K_n}$ are topologically conjugated to models $h_n \in \mathcal{H}$.*

2. Construction of the family of models.

We call a compact set $K \subset D^m := [0,1]^m$ or more generally $K \subset M$ where M is an m-dimensional manifold *a box* if it is homeomorphic to D^m. Models are continuous maps of the $D^m := [0,1]^m$. We will discuss separately the cases $m = 1$ and $m \geq 2$.

Definition 2.1. (Models in the interval.)
Let $h \in C^0([0,1])$. We call h a *model* if $h([0,1]) \subset (0,1)$, and there exists an ergodic h-invariant measure ν such that $h_\nu(h) = +\infty$.

Lemma 2.2. [CT, Theorem 39]**(Existence of models $m = 1$)**
The family of models in the interval is a dense G_δ-set in $C^0([0,1])$.

2.1. **Models in dimension 2 or larger.** In this subsection, we assume $m \geq 2$. We denote by $RHom(D^m)$ the space of relative homeomorphisms $h : D^m \to D^m$ (i.e., f is a homeomorphism onto its image included in D^m), with the topology induced by:

$$\|h - g\|_{RHom} := \max\left\{\|h - g\|_{C^0(D^m)},\ \|h^{-1} - g^{-1}\|_{C^0(h(D^m)\cap g(D^m))}\right\}.$$

Definition 2.3. (h-relation from a box to another).
Let $h \in C^0(D^m)$. Let $B, C \subset \text{int}(D^m)$ be two boxes. We write

$$B \xrightarrow{h} C \text{ if } h(B) \cap \text{int}(C) \neq \emptyset.$$

Observe that this condition is open in $C^0(D^m)$ and also in $RHom(D^m)$.

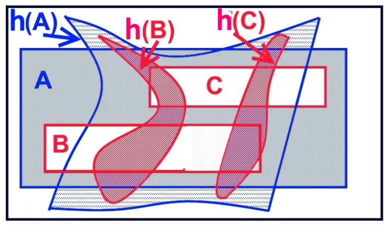

FIGURE 1. An atom A of generation 0 and two atoms B, C of generation 1 for a map f of D^2.

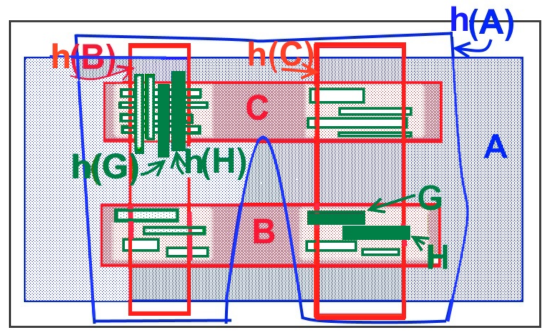

FIGURE 2. An atom A of generation 0, two atoms B, C of generation 1, and 16 atoms of generation 2.

Definition 2.4. (Atoms of generation 0 and 1) *(See Figure 1)*
We call a box $A \subset \text{int}(D^m)$ *an atom of generation 0* for h, if there exists two disjoint boxes $B_1, B_2 \subset \text{int}(A)$, which we call *atoms of generation 1* such that

$$B_i \xrightarrow{h} B_j \quad \forall\, i, j \in \{1, 2\}.$$

If A is an atom of generation 0 and B_1, B_2 are the two atoms of generation 1, we denote $\mathcal{A}_0 := \{A\}$, $\mathcal{A}_1 := \{B_1, B_2\}$.

Definition 2.5. Atoms of generation $\mathbf{n \geq 2}$ *(See Figure 2)*
Assume by induction that the finite families $\mathcal{A}_0, \mathcal{A}_1 \ldots, \mathcal{A}_{n-1}$ of atoms for $h \in C^0(D^m)$ of generations up to $n-1$ are already defined, such that the atoms of the same generation $j = 1, \ldots, n-1$ are pairwise disjoint, contained in the interior of the $(j-1)$-atoms in such a way that all the $(j-1)$-atoms contain the same number of j-atoms and

$$\#\mathcal{A}_j = 2^{j^2} \quad \forall\, j = 0, 1, \ldots, n-1.$$

Assume also that for all $j \in \{1, \ldots, n-1\}$ and for all $B \in \mathcal{A}_j$:

$$\#\{C \in \mathcal{A}_j \colon B \xrightarrow{h} C\} = 2^j, \qquad \#\{D \in \mathcal{A}_j \colon D \xrightarrow{h} B\} = 2^j.$$

Denote

$$\mathcal{A}_j^{2^*} := \{(B, C) \in \mathcal{A}_j^2 : \quad B \xrightarrow{h} C\},$$

$$\mathcal{A}_j^{3^*} := \{(D, B, C) \in \mathcal{A}_j^3 : \quad D \to_h B, \quad B \xrightarrow{h} C\}.$$

We call the boxes of a finite collection $\mathcal{A}_n$ of pairwise disjoint boxes, *atoms of generation n*, or *n-atoms* if they satisfy the following conditions (see Figure 2):

• Each atom of generation n is contained in the interior of an atom of generation $n-1$, and the interior of each atom $B \in \mathcal{A}_{n-1}$ contains exactly $2^n \cdot 2^{n-1} = 2^{2n-1}$ pairwise disjoint n-atoms, which we call *the children of B.* We denote

$$\Omega_n(B) := \{G \in \mathcal{A}_n \colon G \subset \text{int}(B)\}, \quad \#\Omega_n(B) = 2^{2n-1} \ \ \forall\ B \in \mathcal{A}_{n-1}.$$

Then,

$$\mathcal{A}_n = \overset{\circ}{\bigcup_{B \in \mathcal{A}_{n-1}}} \Omega_n(B),$$

$$\#\mathcal{A}_n = (\#\mathcal{A}_{n-1})(\#\Omega_n(B)) = 2^{(n-1)^2} \cdot 2^{2n-1} = 2^{n^2}.$$

• For each pair $(D,B) \in \mathcal{A}^{2^*}_{n-1}$, the collection of children of B is partitioned in 2^{n-1} sub-collections $\Omega_n(D,B)$ as follows:

$$\Omega_n(D,B) := \Big\{G \in \mathcal{A}_n \colon G \subset \text{int}(B), \quad D \overset{h}{\to} G\Big\} \ \ \forall\ (D,B) \in \mathcal{A}^{2^*}_{n-1},$$

$$\#\Omega_n(D,B) = 2^n.$$

Then

$$\Omega_n(B) = \overset{\circ}{\bigcup_{D \colon (D,B) \in \mathcal{A}^{2^*}_{n-1}}} \Omega_n(D,B) \ \ \forall\ B \in \mathcal{A}_{n-1},$$

$$\#\Omega_n(B) = \big(\#\{D \in \mathcal{A}_{n-1} \colon D \to_h B\}\big) \cdot \big(\#\Omega_n(D,B)\big) =$$
$$2^{n-1} \cdot 2^n = 2^{2n-1} \ \ \forall\ B \in \mathcal{A}_{n-1}.$$

• For each 3-uple $(D,B,C) \in \mathcal{A}^{3^*}_{n-1}$ there exists exactly 2 n-atoms, say G_1, G_2 such that

$$\{G_1, G_2\} = \Gamma_n(D,B,C) := \{G \in \Omega_n(D,B) \subset \mathcal{A}_n \colon G \overset{h}{\to} C\},$$

$$\#\Gamma_n(D,B,C) = 2.$$

For example, in Figure 2 we have $\{F,G\} = \Gamma_2(C,B,C)$.

Besides, we assume the following properties for all $G \in \Gamma_n(D,B,C)$

$$h(G) \cap C' = \emptyset \ \forall\ C' \in \mathcal{A}_{n-1} \text{ such that } C' \neq C,$$

$$\text{and} \ \ G \overset{h}{\to} E \ \ \forall\ E \in \Omega_n(C).$$

From the above conditions we deduce:

$$\Omega_n(D,B) = \overset{\circ}{\bigcup_{C \colon (B,C) \in \mathcal{A}^{2^*}_{n-1}}} \Gamma_n(D,B,C);$$

$$\mathcal{A}_n = \overset{\circ}{\bigcup_{(D,B,C) \in \mathcal{A}^{3^*}_{n-1}}} \Gamma_n(D,B,C);$$

and for any pair $(G, E) \in \mathcal{A}_n^2$:

$$G \to_h E \text{ if and only if } \exists\ (D, B, C) \in \mathcal{A}_{n-1}^{3^*} \text{ such that}$$
$$G \in \Gamma_n(D, B, C),\ E \in \Omega_n(C),$$
$$G \not\to_h^n E \quad \text{if and only if} \quad h(G) \cap \operatorname{int}(E) = \emptyset.$$

We also deduce the following properties for any atom $G \in \mathcal{A}_n$:

$$\#\{E \in \mathcal{A}_n \colon G \to_h E\} = 2^n, \qquad \#\{E \in \mathcal{A}_n \colon E \to_h G\} = 2^n.$$

In fact, on the one hand for each atom $G \in \mathcal{A}_n$, the number of atoms $E \in \mathcal{A}_n$ such that $G \to_h E$ equals the number $\#\Omega_n(B, C) = 2^n$, where B and C are the unique $(n-1)$-atoms such that $G \in \Gamma_n(D, B, C)$ for some $D \in \mathcal{A}_{n-1}$. On the other hand, the number of atoms $E \in \mathcal{A}_n$ such that $E \to_h G$ equals

$$(\#\Gamma_n(D, B, C)) \cdot (\#\{D \in \mathcal{A}_{n-1} :\ D \to_h B) = 2 \cdot 2^{n-1} = 2^n,$$

where B and C are the unique $(n-1)$-atoms such that $E \in \Omega_n(B, C)$.

Since $2^n < 2^{n^2}$ if $n > 1$, we conclude that not all the pairs (G, E) of atoms of generation $n \geq 2$ satisfy $G \xrightarrow{h} E$.

Remark. The above conditions are open in $C^0(D^m)$ and in $RHom(D^m)$.

Definition 2.6. (Models in dimension 2 or larger)
We call $h \in C^0(D^m)$ *a model* if $h(D^m) \subset \operatorname{int}(D^m)$, and there exists a sequence $\{\mathcal{A}_n\}_{n \geq 0}$ of finite families of pairwise disjoint boxes contained in $\operatorname{int}(D^m)$ that are atoms of generation $0, 1, \ldots, n, \ldots$ respectively for h (according to Definitions 2.4, and 2.5) such that

$$\lim_{n \to +\infty} \max_{A \in \mathcal{A}_n} \operatorname{diam} A = 0. \tag{1}$$

Denote by $\mathcal{H}$ the family of all the models in $C^0(D^m)$.

Remark 2.7. If $m \geq 2$, the family $\mathcal{H}$ is a non-empty G_δ-set in $C^0(D^m)$ and $\mathcal{H} \cap RHom(D^m)$ is a non-empty G_δ-set in $RHom(D^m)$. In fact, as remarked at the end of Definition 2.5, for each $n \geq 1$ the family $\mathcal{H}_n$ of maps that have atoms up to generation n, is open in $C^0(D^m)$ and also in $RHom(D^m)$. Besides, the conditions $h(D^m) \subset \operatorname{int}(D^m)$ and $\max_{A \in \mathcal{A}_n} \operatorname{diam}(A) < \epsilon_n$ are also open.

2.2. **Paths of atoms.**

Definition 2.8. Let $h \in \mathcal{H} \subset C^0(D^m)$, $l \geq 2$ and let $(A_1, A_2, \ldots, A_l)$ be a l-uple of atoms for h of the same generation n, such that

$$A_i \xrightarrow{h} A_{i+1} \quad \forall\ i \in \{1, 2, \ldots, l-1\}.$$

We call $(A_1, A_2, \ldots, A_l)$ an *l-path of n-atoms from A_1 to A_l*. Let $\mathcal{A}_n^{l^*}$ denote the family of all the l-paths of atoms of generation l.

Lemma 2.9. *For all $n \geq 1$, and for all $A_1, A_2 \in \mathcal{A}_n$ there exists $l \geq 1$ and an l-path of n-atoms from A_1 to A_2.*

Proof. For $n = 1$, the result is trivial with $l = 1$ (see Definition 2.4). Let us assume by induction that the result holds for some $n - 1 \geq 1$ and let us prove it for n.

Let $E, F \in \mathcal{A}_n$. From Definition 2.5, there exists unique atoms $B_{-1}, B_0, B_1 \in \mathcal{A}_{n-1}$ such that $E \in \Gamma_n(B_{-1}, B_0, B_1)$. Then $B_{-1} \stackrel{h}{\to} B_0$, $E \subset B_0$ and

$$E \stackrel{h}{\to} E_1 \quad \forall\, E_1 \in \Omega_n(B_0, B_1). \tag{2}$$

Analogously, there exists unique atoms $B_*, B_{*+1} \in \mathcal{A}_{n-1}$ such that $F \in \Omega_n(B_*, B_{*+1})$. Then $B_* \stackrel{h}{\to} B_{*+1}$, $F \subset B_{*+1}$ and

$$E_* \stackrel{h}{\to} F \quad \forall\, E_* \in \bigcup_{\substack{B_{*-1} \in \mathcal{A}_{n-1}: \\ B_{*-1} \stackrel{h}{\to} B_*}} \Gamma_n(B_{*-1}, B_*, B_{*+1}) \tag{3}$$

Since $B_1, B_* \in \mathcal{A}_{n-1}$ the induction hypothesis ensures that there exists l and an l-path $(B_1, B_2, \dots, B_l)$ from B_1 to $B_l = B_*$. We write $B_{*-1} = B_{l-1}$, $B_* = B_l$, $B_{*+1} = B_{l+1}$. So Assertion (3) becomes

$$E_l \stackrel{h}{\to} F \quad \forall\, E_l \in \Gamma_n(B_{l-1}, B_l, B_{l+1}) \tag{4}$$

Taking into account that $B_{i-1} \;\to_h B_i$ for $1 < i \leq l$, and applying Definition 2.5, we deduce that if $E_{i-1} \in \Gamma_n(B_{i-2}, B_{i-1}, B_i) \subset \mathcal{A}_n$, then

$$E_{i-1} \to_h E_i \quad \forall\, E_i \in \Omega_n(B_{i-1}, B_i) \quad \forall\, 1 < i \leq l. \tag{5}$$

Finally, joining Assertions (2), (4) and (5) we obtain an $(l+2)$-path $(E, E_1, \dots, E_l, F)$ from E to F, as wanted. □

Lemma 2.10. *Let $n, l \geq 2$. For each l-path $(B_1, \dots, B_l)$ of $(n-1)$-atoms there exists an l- path $(E_1, E_2, \dots, E_l)$ of atoms of generation n such that $E_i \subset int(B_i)$ for all $i = 1, 2, \dots, l$.*

Proof. In the proof of Lemma 2.9 for each l-path $(B_1, B_2, \dots, B_l)$ of $(n-1)$-atoms we have constructed the l-path $(E_1, E_2, \dots, E_l)$ of n-atoms as wanted. □

2.3. **Construction of models in dimension larger than 1.** Lemma 2.2 states that the family $\mathcal{H}$ of models is a nonempty G_δ-set in $C^0([0,1])$. In the case $m \geq 2$, we will prove the following result, whose difficult part is b). The difficulty would not be so, if we were only constructing models h. For further uses in Section 4, we need h to be besides isotopic to an arbitrarily given homeomorphism $f \in RHom(D^m)$.

Lemma 2.11. *If $m \geq 2$, then*
a) *The family $\mathcal{H}$ of models is a nonempty G_δ-set in $C^0(D^m)$ and $\mathcal{H} \cap RHom(D^m)$ is a nonempty G_δ-set in $RHom(D^m)$.*
b) *For all $f \in RHom(D^m)$ such that $f(D^m) \subset int(D^m)$, there exists $\psi \in Hom(D^m)$ and $h \in \mathcal{H} \cap RHom(D^m)$ such that*

$$\psi|_{\partial D^m} = \mathrm{id}|_{\partial D^m}, \quad \textit{and} \quad \psi \circ f = h.$$

Proof. Taking into account Remark 2.7, to prove Lemma 2.11 it is enough to prove part b).

We will divide the construction of ψ and h into several steps:

Step 1. Construction of the atoms of generation 0 and 1.

Since $f(D^m) \subset \mathrm{int}(D^m)$, there exists a box $A_0 \subset \mathrm{int}(D^m)$ such that $f(D^m) \subset \mathrm{int}(A_0)$. The box A_0 is the *atom of generation 0 or 0-atom* for a map $h_1 \in RHom(D^m)$ that we construct in this step. Let $\mathcal{A}_0 := \{A_0\}$. We choose two pairwise disjoint boxes B_1 and B_2 contained in $\mathrm{int}(A_0)$. They are the atoms of generation 1, or for short 1-atoms for h_1. Let $\mathcal{A}_1 := \{B_1, B_2\}$. We choose them small enough such that

$$\max_{B \in \mathcal{A}_1} \max\{\mathrm{diam}(B),\ \mathrm{diam}(f(B))\} < \frac{1}{2}. \tag{6}$$

Let K be an m-dimensional box ($m \geq 2$) and consider any two finite sets $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ of points in $\mathrm{int}(K)$. A simple proof by induction on k shows that there exists a homeomorphism $\psi : K \to K$ which is the identity on ∂K and satisfies $\psi(x_i) = y_i$. We apply this result to the box A_0 as follows. For each $B \in \mathcal{A}_1$ we choose two different points $\widetilde{e}_i(B) \in \mathrm{int}(B)$, $\ i = 1, 2$. Then there exists a homeomorphism $\widetilde{\psi}_1 : A_0 \to A_0$ such that

$$\widetilde{\psi}_1|_{\partial A_0} = \mathrm{id}|_{\partial A_0}, \ \ \widetilde{\psi}_1(f(\widetilde{e}_i(B_j)) = \widetilde{e}_j(B_i) \ \ \forall\, i, j \in \{1, 2\}.$$

Now, we extend $\widetilde{\psi}_1$ to be a homeomorphism of the whole box D^m by defining $\widetilde{\psi}_1(x) = x \quad \forall\, x \in D^m \setminus A_0$. In particular

$$\widetilde{\psi}_1|_{\partial D^m} = \mathrm{id}|_{\partial D^m}.$$

Define

$$\widetilde{h}_1 := \widetilde{\psi}_1 \circ f.$$

Recalling Definitions 2.3 and 2.4, we deduce that $B_i \stackrel{\widetilde{h}_1}{\to} B_j$ for all $i, j \in \{1, 2\}$. So A_0 is an atom of generation 0 for $\widetilde{h}_1$, and B_1, B_2 are its atoms of generation 1.

Nevertheless, we will not use the homeomorphisms $\widetilde{h}_1$ and $\widetilde{\psi}_1$ as they are. We will modify them to obtain new homeomorphisms h_1 and ψ_1 such that for any ordered 3-uple $(D, B, C) \in \mathcal{A}_1$ the interior of the intersection $h_1(D) \cap B \cap h_1^{-1}(C)$ is nonempty. This additional property will allow us to construct later the atoms of generation 2 for a new homeomorphism h_2 without changing too much the previous homeomorphism h_1.

We will modify $\widetilde{\psi}_1$ only in the interiors of the boxes $f(B), B \in \mathcal{A}_1$, to obtain a new homeomorphism ψ_1 such that $\psi_1(f(B)) = \widetilde{\psi}_1(f(B)) = \widetilde{h}_1(B)$ for all $B \in \mathcal{A}_1$. Therefore, the same atom of generation 0 and the same two atoms of generation 1 for $\widetilde{h}_1$, will still be atoms of generations 0 and 1 for the new homeomorphism defined by $h_1 := \psi_1 \circ f$.

From the construction of $\widetilde{\psi}$ and $\widetilde{h}_1$, for any ordered pair $(D, B) \in \mathcal{A}_1$ there exists a point $\widetilde{e}_i(B) \in \mathrm{int}(B \cap \widetilde{h}_1(D)) \neq \emptyset$. Denote by

$$S(D, B)$$

the connected component of $(B \cap \widetilde{h}_1(D))$ such that $\widetilde{e}_i(B) \in \mathrm{int} S(D, B)$. Choose two different points

$$e_i(D, B) \in \mathrm{int}(S(D, B)), \quad i = 1, 2,$$

and a (one-to-one and surjective) permutation θ_1 of the finite set

$$L_1 := \{e_i(D, B) \in \mathrm{int}(S(D, B)) : \quad (D, B) \in (\mathcal{A}_1)^2, \quad i = 1, 2\} \tag{7}$$

such that

$$\theta_1(e_i(D, B)) = e_k(B, C)$$

for some $C \in \mathcal{A}_1$ and some $k \in \{1, 2\}$, in such a way that for each fixed ordered 3-uple $(D, B, C) \in (\mathcal{A}_1)^3$ there exists one and only one point $e_i(D, B)$ such that $\theta_1(e_i(D, B)) = e_k(B, C)$.

Now, for each $B \in \mathcal{A}_1$ we construct a homeomorphism

$$\psi_1 : f(B) \to \widetilde{\psi}_1(f(B)) = \widetilde{h}_1(B)$$

such that

$$\psi_1|_{\partial f(B)} = \widetilde{\psi}_1|_{\partial f(B)}, \qquad \psi_1(f(e_i(D, B)) = \theta_1(e_i(D, B)) \;\; \forall \; D \in \mathcal{A}_1.$$

Such ψ_1 exists as the composition of $\widetilde{\psi}_1$ with a homeomorphism that leaves fixed the points of $\partial(\widetilde{\psi}_1 \circ f(B))$ for each both atoms $B \in \mathcal{A}_1$, and transforms each point $\widetilde{\psi}_1 \circ f(e_i(D, B))$ on $\theta_1(e_i(D, B))$.

Now, we extend ψ_1 to the whole box D^m by defining $\psi_1(x) = \widetilde{\psi}_1(x) \;\; \forall \, x \in D^m \setminus \bigcup_{B \in \mathcal{A}_1} f(B)$. In particular

$$\psi_1|_{\partial D^m} = \widetilde{\psi}_1|_{\partial D^m} = \mathrm{id}|_{\partial D^m}.$$

Define

$$h_1 := \psi_1 \circ f.$$

As mentioned at the beginning, the property $h_1(B) = \widetilde{h}_1(B)$ for both atoms $B \in \mathcal{A}_1$ implies that they are also atoms of generation 1 for h_1. But now, they have the following additional property:

There exists a one-to-one correspondence between the 3-uples $(D, B, C) \in (\mathcal{A}_1)^3$ and the eight points of the set L_1 in Equality (7), such that

$$e(D,B,C) := e_i(D,B) \in \operatorname{int}\big(S(D,B) \cap h_1^{-1}(S(B,C))\big). \tag{8}$$

Hence $e(D,B,C) \in \operatorname{int}\big(h_1(D) \cap B \cap h_1^{-1}(C)\big)$.

Step 2. Construction of the atoms of generation n.

Assume by induction that we have constructed the families $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ of atoms up to generation n for $h_n = \psi_n \circ f$, where $\psi_n \in Hom(D^m)$ is a homeomorphism that leaves fixed the points of ∂D^m, and such that

$$\max_{B \in \mathcal{A}_n} \max\{\operatorname{diam}(B), \operatorname{diam}(f(B))\} < \frac{1}{2^n}. \tag{9}$$

Assume also the following two assertions for all $i = 1, \ldots, n$:

- $h_i(x) = h_{i-1}(x) \ \ \forall\, x \in D^m \setminus \bigcup_{B \in \mathcal{A}_{i-1}} B$;
- for all $(D,B,C) \in \mathcal{A}_n^{3^*}$ there exists a point $e(D,B,C)$ such that

$$L_n := \{e(D,B,C) \colon (D,B,C) \in \mathcal{A}_n^{3*}\}$$

is h_n-invariant, and

$$e(D,B,C) \in \operatorname{int}\big(S(D,B) \cap h_n^{-1}(S(B,C))\big), \tag{10}$$

where $S(D,B)$ and $S(B,C)$ are fixed connected components of $B \cap h_n(D)$ and of $C \cap h_n(B)$ respectively.

Let us construct the family $\mathcal{A}_{n+1}$ of atoms of generation $n+1$ and the homeomorphisms h_{n+1} and ψ_{n+1}. First, for each $(D,B) \in (\mathcal{A}_n)^{*2}$ we choose a box $R(D,B)$ such that

$$e(D,B,C) \in \operatorname{int}(R(D,B)) \subset \operatorname{int}\big(S(D,B)\big) \tag{11}$$

$$\forall\, C \in \mathcal{A}_n \text{ such that } B \xrightarrow{h_n} C.$$

Note that $e(D,B,C) \in L_n$, the set L_n is h_n-invariant, and besides $e(D,B,C) \in \operatorname{int}(h_n^{-1}(S(B,C)))$, thus

$$e(D,B,C) \in \operatorname{int}\Big(R(D,B) \cap h_n^{-1}(R(B,C))\Big) \neq \emptyset.$$

Then,

$$h_n(e(D,B,C)) = e(B,C,\cdot) \in \operatorname{int}(R(B,C)).$$

Next, for each $(D,B,C) \in \mathcal{A}_n^{3*}$ we choose two pairwise disjoint boxes G_1, G_2 contained in the interior of $R(D,B) \cap h_n^{-1}(R(B,C))$, satisfying

$$\max\{\operatorname{diam}(G_i), \operatorname{diam}(f(G_i))\} < \frac{1}{2^{n+1}}. \tag{12}$$

Denote

$$\Gamma_{n+1}(D,B,C) := \{G_1, G_2\},$$

$$\Omega_{n+1}(D,B) := \mathop{\overset{\circ}{\bigcup}}_{\substack{C \in \mathcal{A}_n : \\ B \overset{h}{\to} C}} \Gamma_{n+1}(D,B,C),$$

$$\mathcal{A}_{n+1} := \mathop{\overset{\circ}{\bigcup}}_{(D,B) \in \mathcal{A}_n^{2*}} \Omega_{n+1}(D,B) = \mathop{\overset{\circ}{\bigcup}}_{(D,B,C) \in \mathcal{A}_n^{3*}} \Gamma_{n+1}(D,B,C).$$

Note that for all $G \in \mathcal{A}_{n+1}$,

$$G \in \Omega_{n+1}(D,B) \text{ if and only if } G \subset \operatorname{int}(R(D,B)).$$

The boxes of the family $\mathcal{A}_{n+1}$ will be the $(n+1)$-atoms of a new homeomorphism h_{n+1} that we construct as follows.

In the interior of each box $G \in \mathcal{A}_{n+1}$ we choose 2^{n+1} different points $\widetilde{e}_i(G), i = 1, 2 \ldots, 2^{n+1}$, and denote

$$\widetilde{L}_{n+1} := \{\widetilde{e}_i(G) \colon G \in \mathcal{A}_n,\ 1 \le i \le 2^n\}.$$

Then, we build a permutation $\widetilde{\theta}$ of the set $\widetilde{L}_{n+1}$ such that for all $(D,B,C) \in \mathcal{A}_n^{3^*}$, and for each box $G \in \Gamma_{n+1}(D,B,C)$, it transforms the point $\widetilde{e}_i(G) \subset \operatorname{int}(G) \subset \operatorname{int}(R(D,B))$ into the point

$$\widetilde{\theta}(\widetilde{e}_i(G)) = \widetilde{e}_k(E) \in \operatorname{int}(R(B,C))$$

for some $k \in \{1, \ldots, 2^{n+1}\}$ and for some box $E = E_i \in \Omega_{n+1}(B,C)$ satisfying $E_i \neq E_j$ if $i \neq j$.

We consider $\widetilde{\psi}_{n+1}$ defined by the following constraints. For each $(B,C) \in \mathcal{A}_n^{2*}$ the homeomorphism

$$\widetilde{\psi}_{n+1} : \psi_n^{-1}(R(B,C)) \to R(B,C)$$

satisfies

$$\widetilde{\psi}_{n+1}(x) = \psi_n(x) \ \ \forall\ x \in \psi_n^{-1}(\partial R(B,C)) = f \circ h_n^{-1}(\partial R(B,C)),$$

$$\widetilde{\psi}_{n+1}(f(e)) = \widetilde{\theta}(e) \ \ \forall\ e \in \widetilde{L}_{n+1} \cap \operatorname{int}(h_n^{-1}(R(B,C))).$$

Such homeomorphism $\widetilde{\psi}_{n+1}$ exists because the finite set

$$\{f(e) : \quad e \in \widetilde{L}_{n+1} \cap \operatorname{int}(h_n^{-1}(R(B,C)))\}$$

is contained in the interior of $f \circ h_n^{-1}(R(B,C)) = \psi_n^{-1}(R(B,C)))$, and coincides with

$$\{f(e_i(G))\colon G \in \Gamma_{n+1}(D,B,C) \text{ for some } D \in \mathcal{A}_n,\ i = 1,\dots,2^{n+1}\}.$$

So, the corresponding images $\widetilde{\psi}_{n+1}(f(e)) = \widetilde{\theta}(e)$ are the points of the finite set

$$\{\widetilde{\theta}(e_i(G))\colon G \in \Gamma_{n+1}(D,B,C) \quad \text{for some } D \in \mathcal{A}_n\} =$$

$$\{e_k(E)\colon E \in \Omega_{n+1}(B,C)\} = \widetilde{L}_{n+1} \cap R(B,C),$$

which is contained in the interior of $R(B,C))$.

After extending $\widetilde{\psi}_{n+1}$ to the whole cube D^m by putting it equal to ψ_n in the complement of $\bigcup_{(B,C)\in\mathcal{A}_n^{2*}} \psi_n^{-1}(R(B,C))$, we define

$$\widetilde{h}_{n+1} := \widetilde{\psi}_{n+1} \circ f.$$

Since $\widetilde{h}_{n+1}(B) = h_n(B) \ \ \forall\ B \in \mathcal{A}_n$, the same atoms up to generation n for h_n are still atoms up to generation n for $\widetilde{h}_{n+1}$. But besides, the boxes of the family $\mathcal{A}_{n+1}$ are now (n_1)-atoms for $\widetilde{h}_{n+1}$.

Nevertheless, we will not adopt the homeomorphisms $\widetilde{h}_{n+1}$ and $\widetilde{\psi}_{n+1}$ as they are. We will modify them to obtain new homeomorphisms h_{n+1} and ψ_{n+1} such that Assertion (10) also holds for $n+1$ instead of n. Precisely, modifying $\widetilde{\psi}_{n+1}$ only in the interiors of the boxes $f(G)$ for all the atoms $G \in \mathcal{A}_{n+1}$, we construct a new homeomorphism ψ_{n+1} such that $h_{n+1} := \psi_{n+1} \circ f$ has the same atoms up to generation $n+1$ than $\widetilde{h}_{n+1}$. Let us describe how to build ψ_{n+1}:

From the construction of $\widetilde{\psi}_{n+1}$ and $\widetilde{h}_{n+1}$, for all $(G,E) \in \mathcal{A}_{n+1}^{2*}$ there exists one and only one point $\widetilde{e}_i(G) \in \operatorname{int}(G)$, and one and only one point $\widetilde{e}_k(E)$, such that

$$\widetilde{h}_{n+1}(\widetilde{e}_i(G)) = \widetilde{\psi}_{n+1} \circ f(\widetilde{e}_i(G)) = e_k(E) \in \operatorname{int}(E).$$

Therefore

$$\widetilde{e}_k(E) \in \operatorname{int}(E \cap \widetilde{h}_{n+1}(G)) \neq \emptyset.$$

Denote by

$$S(G,E)$$

the connected component of $E \cap \widetilde{h}_{n+1}(G)$ that contains the point $\widetilde{e}_k(E)$. Choose 2^{n+1} different points

$$e_i(G,E) \in \operatorname{int}(S(G,E)), \quad i = 1,\dots,2^{n+1}$$

and a permutation θ of the finite set

$$L_{n+1} := \{e_i(G,E) : \quad (G,E) \in \mathcal{A}_{n+1}^{2^*}, \quad i = 1,\dots,2^{n+1}\} \tag{13}$$

such that for each fixed $(G, E, F) \in (\mathcal{A}_{n+1})^{3^*}$ there exists a unique point $e_i(G, E)$, and a unique point $e_k(E, F)$, satisfying

$$\theta(e_i(G, E)) = e_k(E, F).$$

Then, for each $G \in \mathcal{A}_{n+1}$ construct a homeomorphism

$$\psi_{n+1} : f(G) \to \widetilde{\psi}_{n+1}(f(G)) = \widetilde{h}_{n+1}(G)$$

such that

$$\psi_{n+1}|_{\partial f(G)} = \widetilde{\psi}_{n+1}|_{\partial f(G)},$$
$$\psi_{n+1}(f(e_i(G, E)) = \theta(e_i(G, E))$$
$$\forall\ E \in \mathcal{A}_{n+1} \text{ such that } G \xrightarrow{h} E, \quad \forall\ i = 1, \ldots, 2^{n+1}.$$

Finally, extend ψ_{n+1} to the whole box D^m by defining $\psi_{n+1}(x) = \widetilde{\psi}_{n+1}(x) \quad \forall\ x \in D^m \setminus \bigcup_{G \in \mathcal{A}_{n+1}} f(G)$. In particular

$$\psi_{n+1}|_{\partial D^m} = \widetilde{\psi}_{n+1}|_{\partial D^m} = \mathrm{id}|_{\partial D^m}.$$

Define

$$h_{n+1} := \psi_{n+1} \circ f.$$

As said above, the property $h_{n+1}(G) = \widetilde{h}_{n+1}(G)$ for all the atoms $G \in \bigcup_{j=0}^{n+1} \mathcal{A}_j$ implies that the boxes of the families $\mathcal{A}_0, \ldots, \mathcal{A}_{n+1}$ are also atoms up to generation $n+1$ for h_{n+1}. But now, they have the following additional property: there exists a one-to-one correspondence between the 3-uples $(G, E, F) \in (\mathcal{A}_{n+1})^{3^*}$ and the points of the set L_{n+1} of Equality (13), such that

$$e(G, E, F) := e_i(G, E) \in \mathrm{int}\big(S(G, E) \cap h_{n+1}^{-1}(S(E, F))\big), \tag{14}$$

where $S(G, E)$ and $S(E, F)$ are fixed connected components of $E \cap h_n(G)$ and of $F \cap h_n(E)$ respectively. Therefore, Assertion (10) holds for $n + 1$, and the inductive construction is complete.

Besides, from the above construction we have:

$$\psi_{n+1}(x) = \widetilde{\psi}_{n+1}(x) = \psi_n(x) \text{ if } x \notin \bigcup_{B,C} \psi_n^{-1}(R(B, C)) \subset \bigcup_{B} f(B)$$

$$\psi_{n+1} \circ \psi_n^{-1}(R(B, C)) = \widetilde{\psi}_{n+1} \circ \psi_n^{-1}(R(B, C)) =$$
$$\psi_n \circ \psi_n^{-1}(R(B, C)) = R(B, C) \subset C.$$

Therefore,

$$d(\psi_{n+1}^{-1}(x), \psi_n^{-1}(x)) \leq \max_{B \in \mathcal{A}_n} \mathrm{diam}(f(B)) < \frac{1}{2^n} \quad \forall\ x \in D^m;$$

$$d(\psi_{n+1}(x), \psi_n(x)) \leq \max_{C \in \mathcal{A}_n} \mathrm{diam}(C) < \frac{1}{2^n} \quad \forall\ x \in D^m,$$

$$\|\psi_{n+1} - \psi_n\|_{Hom} < \frac{1}{2^n}. \tag{15}$$

Step 3. The limit homeomorphisms

From Inequality (15) we deduce that the sequence ψ_n is Cauchy in $Hom(D^m)$. Therefore, it converges to a homeomorphism ψ. Besides, by construction $\psi_n|_{\partial D^m} = \mathrm{id}|_{\partial D^m}$ for all $n \geq 1$. Then $\psi|_{\partial D^m} = \mathrm{id}|_{\partial D^m}$.

The convergence of ψ_n to ψ in $Hom(D^m)$ implies that $h_n = \psi_n \circ f \in RHom(D^m)$ converges to $h = \psi \circ f \in RHom(D^m)$ as $n \to +\infty$. Since $f(D^m) \subset \mathrm{int}(D^m)$ and $\psi \in Hom(D^m)$, we deduce that $h(D^m) \subset \mathrm{int}(D^m)$. Besides, by construction $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ are families of atoms up to generation n for h_n, and $h_m(x) = h_n(x)$ for all $x \in D^m \setminus \bigcup_{B \in \mathcal{A}_n} B$ and for all $m \geq n$. Since $\lim_m h_m = h$, the boxes of the family $\mathcal{A}_n$ are n-atoms for h for all $n \geq 0$. Finally, from Inequality (9), the diameters of the n-atoms converge uniformly to zero as $n \to +\infty$. Thus h is a model according to Definition 2.6. □

3. Infinite metric entropy of the models.

The purpose of this section is to prove the following Lemma.

Lemma 3.1. (Main Lemma) *Let $\mathcal{H} \subset C^0(D^m)$ be the family of models with $m \geq 2$ (Definition 2.6). Then, for each $h \in \mathcal{H}$ there exists an ergodic, h-invariant measure ν supported on an h-invariant Cantor set $\Lambda \subset D^m$ such that $h_\nu(h) = +\infty$.*

Remark 3.2. Lemma 3.1 holds, in particular, for $\mathcal{H} \cap RHom(D^m)$.

To prove Lemma 3.1 we need the following definition:

Definition 3.3. (The Λ-set)

Let $h \in \mathcal{H} \subset C^0(D^m)$ be a model map. Let $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n, \ldots$ be its sequence of families of atoms. The subset

$$\Lambda := \bigcap_{n \geq 0} \bigcup_{A \in \mathcal{A}_n} A$$

of $\mathrm{int}(D^m)$ is called *the* Λ-*set* of the map h.

From Definitions 2.4 and 2.5, we know that, for each fixed $n \geq 0$, the set $\Lambda_n := \bigcup_{A \in \mathcal{A}_n} A$, is nonempty, compact, and $\mathrm{int}(\Lambda_n) \supset \Lambda_{n+1}$. Therefore, Λ is also nonempty and compact. Besides, Λ_n is composed of a finite number of connected components $A \in \mathcal{A}_n$, which by Definition 2.6, satisfy $\lim_{n \to +\infty} \max_{A \in \mathcal{A}_n} \mathrm{diam} A = 0$. Since $\Lambda := \bigcap_{n \geq 0} \Lambda_n$, we deduce that the Λ-set is *a Cantor set* contained in $\mathrm{int}(D^m)$.

Lemma 3.4. (Dynamical properties of Λ)

a) *The Λ-set of a model map $h \in \mathcal{H}$ is h-invariant, i.e., $h(\Lambda) = \Lambda$.*

b) *The map h restrict to the Λ-set is topologically transitive.*

Proof. **a)** Let $x \in \Lambda$ and let $\{A_n(x)\}_{n\geq 0}$ the unique sequence of atoms such that $x \in A_n(x)$ and $A_n(x) \in \mathcal{A}_n$ for all $n \geq 0$. Then, $h(x) \in h(A_n(x))$ for all $n \geq 0$. From Definition 2.5, for all $n \geq 0$ there exists an atom $B_n \in \mathcal{A}_n$ such that $A_n(x) \stackrel{h}{\to} B_n$. Therefore $h(A_n(x)) \cap B_n \neq \emptyset$. Let d denote the Hausdorff distance between subsets of D^m, we deduce

$$d(h(x), B_n) \leq \operatorname{diam}\big(h(A_n(x))\big) + \operatorname{diam}\big(B_n\big).$$

Besides, Equality (1) and the continuity of h imply

$$\lim_{n\to+\infty} \max\left\{\operatorname{diam}\big(h(A_n(x))\big), \operatorname{diam}\big(B_n\big)\right\} = 0.$$

Then, for all $\epsilon > 0$ there exists $n_0 \geq 0$ such that $d(h(x), B_n) < \epsilon$ for some atom $B_n \in \mathcal{A}_n$ for all $n \geq n_0$. Since any atom of any generation intersects Λ, we deduce that $d(h(x), \Lambda) < \epsilon \quad \forall\ \epsilon > 0$. Since Λ is compact, this implies $h(x) \in \Lambda$. We have proved that $h(\Lambda) \subset \Lambda$.

Now, let us prove the other inclusion. Let $y \in \Lambda$ and let $\{B_n(y)\}_{n\geq 0}$ the unique sequence of atoms such that $y = h(x) \in B_n(y)$ and $B_n(y) \in \mathcal{A}_n$ for all $n \geq 0$. From Definition 2.5, for all $n \geq 0$ there exists an atom $A_n \in \mathcal{A}_n$ such that $A_n \stackrel{h}{\to} B_n(y)$. Therefore $h(A_n) \cap B_n(y) \neq \emptyset$. We deduce that, for all $n \geq 0$, there exists a point $x_n \in A_n \in \mathcal{A}_n$ such that $h(x_n) \in B_n(y)$. Since any atom A_n contains points of Λ, we obtain

$$d(x_n, \Lambda) \leq \operatorname{diam}(A_n) \geq 0 \text{ and } d(h(x_n), y) \leq \ \operatorname{diam}(B_n(y)) \ \ \forall\ n \geq 0.$$

Let x be the limit of a convergent subsequence of $\{x_n\}_{n\geq 0}$, applying Equality (1) and the continuity of h, we deduce that $d(x, \Lambda) = 0$ and $d(h(x),\, y) = 0$. This means that $y = h(x)$ and $x \in \Lambda$. We have proved that $y \in h(\Lambda)$ for all $y \in \Lambda$; namely $\Lambda = h(\Lambda)$, as wanted.

b) Since the intersection of Λ with the atoms of all the generations generates its topology, to prove that Λ is h-transitive it is enough to prove that for any two atoms A_1, A_2 there exists an $l \geq 1$ such that $h^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$. It is not restrictive to assume that A_1 and A_2 are atoms of the same generation n_0 (if not, take n_0 equal to the largest of both generation and substitute A_i by an atom of generation n_0 contained in A_i). Applying Lemma 2.9 there exists $l \geq 1$ and an l-path from A_1 to A_2. So, from Lemma 2.10, for all $m \geq n_0$ there exists atoms $B_{m,1}, B_{m,2}$ and an l-path from $B_{m,1}$ to $B_{m,2}$ (with constant $l \geq 1$) such that

$$B_{n_0,i} = A_i, \quad B_{m+1,i} \subset B_{m,i} \ \ \forall\ m \geq n_0, \quad i = 1, 2.$$

Construct the following two points x_1 and x_2:

$$\{x_i\} = \bigcap_{n\geq n_0} \bigcup_{m\geq n} B_{m,i}, \qquad i = 1, 2.$$

By Definition 3.3, $x_i \in A_i \cap \Lambda$. So, to finish the proof of part b) it is enough to prove that $h^l(x_1) = x_2$.

Fix $l \geq 1$. Since h is uniformly continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $(y_0, y_1, \dots, y_l) \in (D^m)^l$ satisfies $d(h(y_i), y_{i+1}) < \delta$ for $0 \leq i \leq l$, then the points y_0 and y_l satisfy $d(f^l(y_0), y_l) < \epsilon$. We choose δ small enough so that $d(h^l(x), h^l(y)) < \epsilon$ if $d(x, y) < \delta$.

From Equality (1), there exists $m \geq n_0$ such that $\operatorname{diam}(B_{m,i}) < \epsilon$. Since there exists an l-path from $B_{m,1}$ to $B_{m,2}$, there exists a $(y_0, \dots, y_l)$ as in the previous paragraph with $y_0 \in B_{m,1}$ and $y_l \in B_{m,2}$. Thus

$$d(h^l(x_1), x_2) \leq d(h^l(x_1), h^l(y_0)) + d(h^l(y_0), y_l) + d(y_l, x_1)$$

$$< \operatorname{diam}(h^l(B_{m,1})) + \epsilon + \operatorname{diam}(B_{m,2}) < 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain $h^l(x_1) = x_2$, as wanted. $\square$

Lemma 3.5. (Intersection of Λ with l-paths) *Fix $l, n \geq 1$. Then*
a) *For any $G \in \mathcal{A}_{n+l}$, there exists a unique $(l+1)$-path $(A_0, A_1, \dots, A_l)$ of n-atoms such that $G \cap \Lambda \subset \bigcap_{j=0}^{l} h^{-j}(A_j)$.*
b) *For any $(l+1)$-path $(A_0, A_1, \dots, A_l)$ of n-atoms,*

$$\Lambda \cap \bigcap_{j=0}^{l} h^{-j}(A_j) = \overset{\circ}{\bigcup_{G \in \mathcal{F}_{n,l}(\{A_j\})}} G \cap \Lambda, \tag{16}$$

where $\mathcal{F}_{n,l}(\{A_j\}) := \left\{ G \in \mathcal{A}_{n+l} \colon G \cap \Lambda \subset \bigcap_{j=0}^{l} h^{-j}(A_j) \right\}.$
c) *For any $(l+1)$-path $(A_0, A_1, \dots, A_l)$ of n-atoms,*

$$\#\mathcal{F}_{n,l}(\{A_j\}) = \frac{1}{2^{nl}} \cdot \frac{\#\mathcal{A}_{n+l}}{\#\mathcal{A}_n}.$$

Proof. **a)** From Definition 2.5, for any atom G of generation $n+l$ there exist two unique atoms B, C of generation $n+l-1$ such that $B \to_h C$, $G \subset B$ and $G \xrightarrow{h} E$ for all $E \in \Omega_{n+l}(B, C)$. Besides

$$h(G) \cap \operatorname{int}(F) \neq \emptyset \quad \forall\, F \in \mathcal{A}_{n+l} \setminus \Omega_{n+l}(B, C). \tag{17}$$

We claim that

$$h(G \cap \Lambda) \subset \operatorname{int}(C). \tag{18}$$

In fact, since Λ is h-invariant, for any $x \in G \cap \Lambda$, we have $h(x) \in h(G) \cap \Lambda$. Therefore $h(x)$ is in the interior of some atom $E(x)$ of generation $n + l$ (see Definition 2.5). From (17), $E(x) \in \Omega_{n+l}(B, C)$. Thus $E(x) \subset \operatorname{int}(C)$ and $h(x) \in \operatorname{int}(C)$ for all $x \in G \cap \Lambda$ proving (18).

So, there exists $C_1 \in \mathcal{A}_{n+l-1}$ such that $h(G \cap \Lambda) \subset \operatorname{int}(C_1) \cap \Lambda$. Applying the same assertion to C_1 instead of G, we deduce that there exists $C_2 \in \mathcal{A}_{n+l-2}$ such that $h(C_1 \cap \Lambda) \subset \mathrm{int}(C_2) \cap \Lambda$. So, by induction, we construct atoms

$$C_1, C_2, \ldots, C_l \quad \text{such that} \quad C_j \in \mathcal{A}_{n+l-j} \;\; \text{and}$$
$$h^j(G \cap \Lambda) \subset \mathrm{int}(C_j) \cap \Lambda \;\; \forall\, j = 1, \ldots, l.$$

Since any atom of generation larger than n is contained in a unique atom of generation n, there exists $A_0, A_1, \ldots, A_l \in \mathcal{A}_n$ such that $A_0 \supset G$ and $A_i \supset C_i \;\; \forall\, i = 1, \ldots, l$. We obtain

$$h^j(G \cap \Lambda) \subset \mathrm{int}(A_j) \;\; \forall\, j = 0, 1, \ldots, l.$$

$(A_0, A_1, \ldots, A_l)$ is an $(l+1)$-path since $\emptyset \neq h^j(G \cap \Lambda) \subset h(A_{j-1}) \cap \mathrm{int}(A_j)$; hence $A_{j-1} \to hA_j$ for all $j = 1, \ldots, l$. Besides, $G \cap \Lambda \subset h^{-j}(A_j) \;\; \forall\, j = 0, 1, \ldots, l$, proving a).

b) For the $(l+1)$-path $(A_0, A_1, \ldots, A_l)$ of n-atoms, construct $\widetilde{\mathcal{F}}_{n,l}(\{A_j\}) := \{G \in \mathcal{A}_{n+l} \colon G \cap \Lambda \cap \big(\cap_{j=0}^{l} h^{-j}(A_j)\big) \neq \emptyset\}$. From the definitions of the families $\mathcal{F}_{n,l}$ and $\widetilde{\mathcal{F}}_{n,l}$, and taking into account that Λ is contained in the union of $(n+1)$-atoms, we obtain:

$$int\Big(\bigcup_{G \in \mathcal{F}_{n,l}(\{A_j\})} G \cap \Lambda\Big) \;\subset\; \Lambda \cap \Big(\bigcap_{j=0}^{l} h^{-j}(A_j)\Big) \;\subset\; int\Big(\bigcup_{G \in \widetilde{\mathcal{F}}_{n,l}(\{A_j\})} G \cap \Lambda\Big).$$

Therefore, to prove Equality (16) it is enough to prove that $\widetilde{\mathcal{F}}_{n,l}(\{A_j\}) = \mathcal{F}_{n,l}(\{A_j\})$. Trivially $\widetilde{\mathcal{F}}_{n,l}(\{A_j\}) \supset \mathcal{F}_{n,l}(\{A_j\})$. Let us prove the converse inclusion. If $G \in \widetilde{\mathcal{F}}_{n,l}(\{A_j\})$, then by part a) there exists an $(l+1)$-path $\{\widetilde{A}_j\}$ of n-atoms such that $G \subset \bigcap_{j=0}^{l} h^{-j}(\widetilde{A}_j)$. If the two paths $\{\widetilde{A}_j\}$ and $\{A_j\}$ were different, then the sets $\bigcap_{j=0}^{l} h^{-j}(\widetilde{A}_j)$ and $\bigcap_{j=0}^{l} h^{-j}(A_j)$ would be disjoint. But from the condition, $G \in \widetilde{F}_{n,l}$ we know that G intersects the set $\bigcap_{j=0}^{l} h^{-j}(A_j)$. We deduce that $\{\widetilde{A}_j\} = \{A_j\}$, hence $G \subset \bigcap_{j=0}^{l} h^{-j}(A_j)$ and $G \in \mathcal{F}_{n,l}(\{A_j\})$, as wanted.

c) From Assertion a), and taking into account that for two different $(l+1)$-paths of n-atoms, the respective families $\mathcal{F}_{n,l}$ of $(n+1)$-atoms are disjoint, we obtain:

$$\mathcal{A}_{n+l} = int\Big(\bigcup_{\{A_j\} \in \mathcal{A}_n^{(l+1)^*}} \mathcal{F}_{n,1}(\{A_j\})\Big), \tag{19}$$

where $\mathcal{A}_n^{(l+1)^*}$ denotes the set of all the $(l+1)$-paths of n-atoms. From Definition 2.5, the number of atoms of each generation larger than n that are contained in each $A_j \in \mathcal{A}_n$, and also the number of atoms $G \in \mathcal{A}_{n+j}$ such that $G \to A_j$, are constants that depend only on the generations but not on which particular atoms are chosen. Therefore, there exists a constant $k_{n,l}$ such that $\#\mathcal{F}_{n,l}(\{A_j\}) = k_{n,l}$ for all the $(l+1)$-chains $\{A_j\}$ of n-atoms. So, from Equality (19) we obtain:

$$\#\mathcal{A}_{n+l} = (\#\mathcal{A}_n^{(l+1)^*}) \cdot (\#\mathcal{F}_{n,l}(\{A_j\})).$$

Therefore, to prove Assertion c) it is enough to prove that

$$\#\mathcal{A}_n^{(l+1)^*} = 2^{nl} \cdot (\#\mathcal{A}_n). \tag{20}$$

Each $(l+1)$-path $(A_0, A_1, \ldots, A_l)$ of n-atoms is determined by a free choice of the atom $A_0 \in \mathcal{A}_n$, followed by the choice of the atoms $A_j \in \mathcal{A}_n$ such that $A_j \to_h A_{j-1}$ for all $j = 1, \ldots, l$. From Definition 2.5, we know that for any fixed $A \in \mathcal{A}_n$ the number of atoms $B \in \mathcal{A}_n$ such that $B \stackrel{h}{\to} A$ is 2^n. This implies Equality (20), as wanted. $\square$

Proof of Lemma 3.1. Consider the Λ-set of h, according to Definition 3.3 it is a Cantor set and the σ-algebra of its Borel subsets is generated by the family of its intersection with the atoms of all the generations. In this family define the additive pre-measure given by

$$\nu^*(A \cap \Lambda) := \frac{1}{\#\mathcal{A}_n} \quad \forall\ A \in \mathcal{A}_n \quad \forall\ n \geq 0.$$

Since ν^* is a pre-measure defined in a family of sets that generates the Borel σ-algebra of Λ, there exists a unique Borel probability measure ν supported on Λ such that

$$\nu(A \cap \Lambda) := \frac{1}{\#\mathcal{A}_n} \quad \forall\ A \in \mathcal{A}_n \quad \forall\ n \geq 0.$$

We need to prove that ν is h-invariant, ergodic and that its metric entropy is infinite. Fix $m > 1$, the proof is similar in the 1-dimensional case.

To see that ν is h-invariant, it is enough to prove that

$$\nu(C \cap \Lambda) = \nu(h^{-1}(C \cap \Lambda)) \quad \forall\ C \in \mathcal{A}_n \quad \forall\ n \geq 0. \tag{21}$$

In fact, from Definition 2.5, taking into account that Λ is invariant and that any point in Λ belongs to an atom of generation $n+1$, we obtain:

$$h^{-1}(C \cap \Lambda) = int\Big(\bigcup_{\substack{(D,B) \in \mathcal{A}_n^{2^*} \\ B \stackrel{h}{\to} C}} int\Big(\bigcup_{G \in \Gamma_{n+1}(D,B,C)} (G \cap \Lambda)\Big)\Big);$$

$$\begin{aligned} \nu(h^{-1}(C \cap \Lambda)) &= \sum_{\substack{B \in \mathcal{A}_n \\ B \stackrel{h}{\to} C}} \ \sum_{\substack{D \in \mathcal{A}_n \\ D \stackrel{h}{\to} B}} \ \sum_{G \in \Gamma_{n+1}(B,C,D)} \nu(G \cap \Lambda) \\ &= N_C \cdot N_B \cdot (\#\Gamma_{n+1}(B,C,D)) \cdot \frac{1}{\#\mathcal{A}_{n+1}}, \end{aligned} \tag{22}$$

where $N_X := \#\{Y \in \mathcal{A}_n \colon Y \xrightarrow{h} X\}) = 2^n$ for all $X \in \mathcal{A}_n$. Since $\#\Gamma_{n+1}(B, C, D)) = 2$ and $\#\mathcal{A}_{n+1} = 2^{(n+1)^2}$, we conclude

$$\nu(h^{-1}(C \cap \Lambda)) = 2^n \cdot 2^n \cdot 2 \cdot \frac{1}{2^{(n+1)^2}} = \frac{1}{2^{n^2}} = \frac{1}{\#\mathcal{A}_n} = \nu(C \cap \Lambda),$$

proving Equality (21) as wanted.

We turn to the ergodicity of ν. We assert that if E is a Borel set such that $\nu(E) > 0$, then there exists $n \geq 0$ and an atom $A \in \mathcal{A}_n$ such that $A \cap \Lambda \subset E$. Assume by contradiction that every atom of every generation intersects $\Lambda \setminus E$. Then, for any sequence of atoms $\{A_n\}_{n \geq 0}$ such that $A_n \in \mathcal{A}_n$ and $A_{n+1} \subset A_n$ for all $n \geq 0$, the point $\bigcap_{n \geq 0} A_n$ belongs to $\Lambda \setminus E$. But from Definition 3.3, the union of all such intersections is Λ. We deduce that $\Lambda \subset \Lambda \setminus E$. In other words, $E \cap \Lambda = \emptyset$. Since ν is supported on Λ, we conclude that $\nu(E) = 0$, contradicting the hypothesis and proving the assertion.

Now consider an h-invariant Borel set E, i.e., $h^{-1}(E) = E$, such that $\nu(E) > 0$. We must prove that $\nu(E) = 1$. From the previous assertion, there exists an atom B such that $B \cap \Lambda \subset E$. Therefore

$$h^{-l}(B \cap \Lambda) \subset h^{-l}(E) = E \ \ \forall\, l \geq 1.$$

Since h is topologically transitive (Lemma 3.4 b)), for any atom A of any generation, there exists $l \geq 1$ such that $h^{-l}(B \cap \Lambda) \cap (A \cap \Lambda) \neq \emptyset$. Then $E \cap (A \cap \Lambda) \neq \emptyset$ for all $A \in \mathcal{A}_n$ and for all $n \geq 0$. Again applying the assertion, we conclude that the complement of E has zero ν-measure, and thus ν is ergodic.

Now we turn to the entropy estimate. Fix a natural number $n \geq 1$ and consider the partition $\{\mathcal{A}\}_n$ of Λ composed by the n-atoms intersected with Λ. By definition of metric entropy

$$h_\nu(h) := \sup_{\mathcal{P}} h(\mathcal{P}, \nu) \geq h(\mathcal{A}_n, \nu), \text{ where} \tag{23}$$

$$h(\mathcal{A}_n, \nu) := \lim_{l \to +\infty} \frac{1}{l} H\Big(\bigvee_{j=0}^{l} (h^{-j}\mathcal{A}_n), \nu\Big), \tag{24}$$

$$\mathcal{Q}_l := \bigvee_{j=0}^{l} h^{-j}\mathcal{A}_n := \Big\{\bigcap_{j=0}^{l} h^{-j}A_j \cap \Lambda \neq \emptyset : \quad A_j \in \mathcal{A}_n\Big\},$$

$$H(\mathcal{Q}_l) := -\sum_{X \in \mathcal{Q}_j} \nu(X) \log \nu(X). \tag{25}$$

From part b) of Lemma 3.5, for any nonempty piece

$X := \Lambda \bigcap \Big(\bigcap_{j=0}^{l} h^{-j}A_j\Big) \in \mathcal{Q}_j$, we have

$$\nu(X) = \nu\Big(\bigcap_{j=0}^{l} h^{-j}A_j \cap \Lambda\Big) = \sum_{G \in \mathcal{F}_{n,l}(\{A_j\})} \nu(G \cap \Lambda).$$

Since G is an atom of generation $n+l$, $\nu(G \cap \Lambda) = 1/\#\mathcal{A}_{n+l}$. So, applying part c) of Lemma 3.5, we obtain

$$\nu(X) = \frac{\#\mathcal{F}_{n,l}(\{A_j\})}{\#\mathcal{A}_{n+l}} = \frac{1}{2^{nl} \cdot \#\mathcal{A}_n}.$$

Substituting in Equality (25), $H(\mathcal{Q}_l) = \log(\#\mathcal{A}_n) + nl \cdot \log 2$. Finally, substituting in Equalities (23) and (24), we conclude
$h(\mathcal{A}_n, \nu) := \lim_{l \to +\infty} (1/l) H\Big(\mathcal{Q}_l, \nu\Big) = n \log 2$; therefore
$h_\nu(h) \geq h(\mathcal{A}_n, \nu) = n \log 2$, for all $n \geq 1$; and $h_\nu(h) = +\infty$. □

4. Periodic Shrinking Boxes

In this section we will prove Theorems 1 and 3. The proofs are based on the properties of the models proved in the previous sections, and on the existence of the periodic shrinking boxes which we construct here. Until further notice $m \geq 1$.

Definition 4.1. (Periodic shrinking box) Let $f \in C^0(M)$ and $K \subset M$ be a box. We call K *periodic shrinking with period* $p \geq 1$ for f, if $K,\ f(K),\ f^2(K),\ \dots, f^{p-1}(K)$ are pairwise disjoint, and $f^p(K) \subset \operatorname{int}(K)$. If so, we call $f^p|_K : K \to \operatorname{int}(K)$ the *return map.*

Lemma 4.2. *For any $\delta > 0$, there exists an open and dense set of maps $f \in C^0(M)$ that have a periodic shrinking box K with diam$(K) < \delta$.*

Proof. According to Definition 4.1, the same periodic shrinking box K for f is also a periodic shrinking box with the same period for all $g \in C^0(M)$ near enough f, proving the openness assertion.

We turn to the denseness assertion. Let $f \in C^0(M)$ and $\epsilon > 0$. We will construct $g \in C^0(M)$ and a periodic shrinking box K for g with $\operatorname{diam}(K) < \delta$, such that $\|g - f\|_{C^0} < \epsilon$. We suppose $\delta > 0$ is to be smaller than the ϵ-modulus of continuity of f.

By the Krylov-Bogolyubov theorem invariant measures exist, and thus by the Poincaré Lemma, there exists a recurrent point $x_0 \in M$ for f. First assume that $x_0 \notin \partial M$. So, there exists a box $\overline{B} \subset M$ with $\operatorname{diam}(\overline{B}) < \delta$ such that $x_0 \in B = \operatorname{int}(\overline{B})$. Since x_0 is a recurrent point, there exists a minimum natural number $p \geq 1$ such that $f^p(x_0) \in B$. Taking $\overline{B}$ slightly smaller if necessary, we can assume that $f^j(x_0) \notin \overline{B}$ for all $j = 1, 2, \dots, p-1$. So, there exists a small compact box $\overline{U} \subset B$ as in Figure 3, such that $x_0 \in U = \operatorname{int}(\overline{U})$, the sets $\overline{B}, \overline{U}, f(\overline{U}), \dots, f^{p-1}(\overline{U})$ are pairwise disjoint, and $f^p(\overline{U}) \subset B$.

Consider the homeomorphism $\phi : \overline{B} \to [0,1]^m$ defining the box $\overline{B}$. We can suppose that $\overline{U}$ is homotetic to $\overline{B}$ in the sense that ϕ maps $\overline{U}$ to $[a,b]^m \subset [0,1]^m$. Since $\overline{U}, f^p(\overline{U}) \subset \operatorname{int}(\overline{B})$, there exists a box $\overline{K}$ such

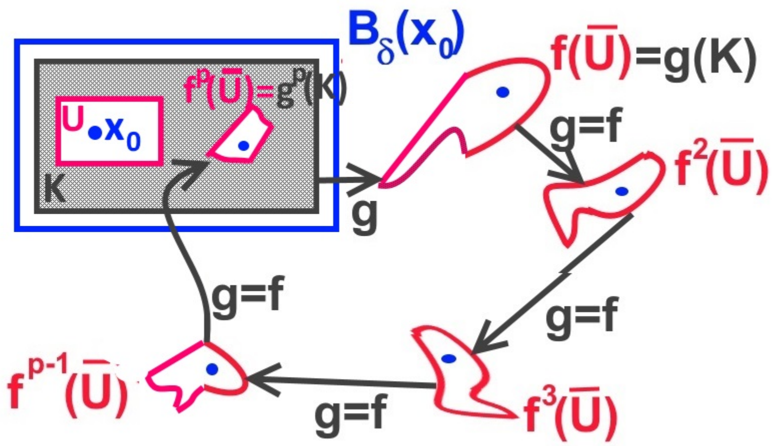

FIGURE 3. Construction of g near f with a periodic shrinking box K for g.

that $\overline{U}, f^p(\overline{U}) \subset \operatorname{int}(\overline{K}) \subset \overline{K} \subset \operatorname{int}(\overline{B})$. In the same sense we choose $\overline{K}$ to be homotetic to $\overline{B}$. Therefore, there exists a homeomorphism $\psi : \overline{B} \to \overline{B}$ such that $\psi(x) = x$ for all $x \in \partial\overline{B}$, and $\psi(K) = \overline{U}$.

Finally, we construct $g \in C^0(M)$ as follows:

$$g(x) := f(x) \ \ \forall\, x \notin \overline{B}, \qquad g(x) = f \circ \psi(x) \ \ \forall\, x \in \overline{B}.$$

By construction, K is a periodic shrinking box of g, by the choice of δ we have $\|g - f\| < \epsilon$.

Now, let us study the case for which M is a manifold with boundary and all the recurrent points of f belong to ∂M. Choose one of such recurrent points $x_0 \in \partial M$. For any $\delta > 0$, there exists a compact box $\overline{B} \subset M$, with $\operatorname{diam}(\overline{B}) \leq \delta$ such that $x_0 \in \partial M \cap \overline{B}$. Since x_0 is recurrent, there exists a smallest natural number $p \geq 1$ such that $f^p(x_0) \in \overline{B}$. But $f^p(x_0)$ is also recurrent. So, $f^p(x_0) \in \partial M \cap \overline{B}$. The previous proof does not work as is. To overcome the problem, we choose a new point $\widetilde{x}_0 \neq x_0$, near enough x_0, such that $\widetilde{x}_0 \in \operatorname{int}(\overline{B}) \setminus \partial M$ and $f^p(\widetilde{x}_0) \in \overline{B}$. Slightly perturbing f if necessary, we can assume that the restriction of f to a small neighborhood of $\widetilde{x}_0$ is a local homeomorphism onto its image. Hence, $f^p(\widetilde{x}_0) \in \operatorname{int}(\overline{B}) \setminus \partial M$. Finally. we repeat the same construction of g and K that we did at the beginning of this proof, but replacing the recurrent point x_0 by $\widetilde{x}_0$. $\square$

Remark 4.3. Note that to obtain the dense property in the proof of Lemma 4.2, we only need to perturb the map f in the interior of the initial box $\overline{B}$ with diameter smaller than δ.

Remark 4.4. We can suppose that p is arbitrarily large by choosing $\overline{B}$ with sufficiently small diameter.

The following lemma is the homeomorphism version of Lemma 4.2. It works for any finite dimension m, including $m = 1$.

Lemma 4.5. *For any $\delta > 0$, there exists an open and dense set of maps $f \in Hom(M)$ that have a periodic shrinking box K with $diam(K) < \delta$.*

Proof. The proof of Lemma 4.2 also works in the case that $f \in \mathrm{Hom}(M)$: in fact, the ϵ-perturbed map g constructed there is a homeomorphism, and to obtain $\|g - f\|_{\mathrm{Hom}(M)} < \epsilon$ it is enough to reduce $\delta > 0$ to be smaller than the ϵ-continuity modulus of f and f^{-1}. □

Remark 4.6. In the proof of Lemmas 4.2 and 4.5, if the starting recurrent point x_0 were a periodic point of period p, then the periodic shrinking box K so constructed would contain x_0 in its interior and have the same period p.

Remark 4.7. For further uses, let us prove that, even if typically a map $f \in C^0(M)$ is not a homeomorphism, we can construct densely in $C^0(M)$ a periodic shrinking box K such that the return map $f^p|_K$ is a homeomorphism onto its image.

Proof. Repeat the beginning of the proof of the density property of Lemma 4.2, up to the construction of the points $x_0, f(x_0), \ldots, f^p(x_0)$ such that $x_0, f^p(x_0) \in B = \mathrm{int}(\overline{B})$ and $f^j(x_0) \notin \overline{B}$. Now, slightly perturb f if necessary, only inside very small open neighborhoods $W_0, W_1, \ldots, W_{p-1}$ of the points $x_0, f(x_0), \ldots, f^{p-1}(x_0)$ respectively, so that $f|_{\overline{W}_i}$ is a homeomorphism onto its image for all $i = 0, 1, \ldots, p-1$. Finally, construct the box $\overline{U}$ as in Figure 3, but small enough so besides $f^j(\overline{U}) \subset W_j$ for all $j = 0, 1, \ldots, p-1$. □

Lemma 4.8. *Let $\delta > 0$. A typical map $f \in C^0(M)$ has a periodic shrinking box K with $diam(K) < \delta$ and such that the return map $f^p|_K$ is topologically conjugated to a model map $h \in \mathcal{H}$* (recall Definition 2.6).

Proof. Let $K \subset M$ be a periodic shrinking box for f. Fix a homeomorphism $\phi : K \to D^m$. The notation $f^p|_K =_\phi h \in \mathcal{H}$, means that $f^p|_K = \phi^{-1} \circ h \circ \phi$ with $h \in \mathcal{H}$.

To prove the G_δ property, assume that $f \in C^0(M)$ has a periodic shrinking box K with $\mathrm{diam}(K) < \delta$, such that $f^p|_K =_\phi h \in \mathcal{H}$. From Definition 4.1, the same box K is also periodic shrinking with period p for all $g \in \mathcal{N}$, where $\mathcal{N} \subset C^0(M)$ is an open neighborhood of f. From Lemma 2.11, $\mathcal{H}$ is a G_δ-set in $C^0(D^m)$, i.e., it is the countable intersection of open families $\mathcal{H}_n \subset C^0(D^m)$. Thus, $f^p|_K =_\phi h \in \mathcal{H}_n$ for

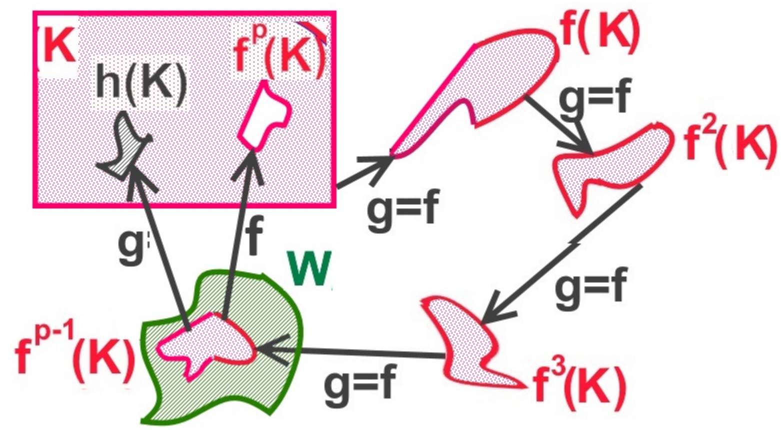

FIGURE 4. Perturbation g of f such that $g^p|_K = h$.

all $n \geq 1$. Since the restriction to K of a continuous map f, and the composition of continuous maps, are continuous operators in $C^0(M)$, we deduce that there exists a sequence of open sets $\mathcal{V}_n \in C^0(M)$ such that

$$g^p|_K =_\phi h \in \mathcal{H} \text{ if } g \in \bigcap_{n\geq 1}^{+\infty}(\mathcal{V}_n \cap \mathcal{N}) \subset C^0(M). \tag{26}$$

In other words, the set of maps $g \in C^0(M)$ that have periodic shrinking box K with $\operatorname{diam}(K) < \delta$, such that the return map $g^p|_K$ coincides, up to a conjugation, with a model map h, contains a G_δ-set in $C^0(M)$.

To show the denseness fix $f \in C^0(M)$ and $\epsilon > 0$. Applying Lemma 4.2 and Remark 4.7, it is not restrictive to assume that f has a periodic shrinking box K with $\operatorname{diam}(K) < \min\{\delta, \epsilon\}$, such that $f^p|_K$ is a homeomorphism onto its image. We will construct $g \in C^0(M)$ to be ϵ-near f and such that $g^p|_K =_\phi h \in \mathcal{H}$.

By Remark 4.4 we can assume that $p \geq 2$. From Definition 4.1 we know that $f^{p-1}(K) \cap K = \emptyset$; thus there exists a box $\overline{W}$ such that $f^{p-1}(K) \subset W := \operatorname{int}(\overline{W})$ and $\overline{W} \cap K = \emptyset$ (Figure 4). Reducing δ if necessary, we can take $\overline{W}$ with an arbitrarily small diameter.

To construct $g \in C^0(M)$ (see Figure 4) let $g(x) := f(x)\ \forall\, x \notin W$ and

$$g(x) := \phi^{-1} \circ h \circ \phi \circ (f^p|_K)^{-1} \circ f(x) \quad \forall\, x \in f^{p-1}(K).$$

This defines a continuous map $g : f^{p-1}(K) \cup (M \setminus W) \to M$ such that $|g(x) - f(x)| < \operatorname{diam}(K) < \epsilon$ for all $x \in f^{p-1}(K) \subset W$ and $g(x) = f(x)$ for all $x \in M \setminus W$. Applying the Tietze Extension Theorem, there exists a continuous extension of g to the whole compact box $\overline{W}$, hence to M, such that $\|g - f\|_{C^0} < \epsilon$. Finally, by construction we obtain

$$g^p|_K = g|_{f^{p-1}(K)} \circ f^{p-1}(K) = \phi^{-1} \circ h \circ \phi \circ (f^p|_K)^{-1} \circ f \circ f^{p-1}(K) =_\phi h,$$

ending the proof of Lemma 4.8. □

Lemma 4.9. *Assume that $m = dim(M) \geq 2$ and let $\delta > 0$. A typical homeomorphism $f \in Hom(M)$ has a periodic shrinking box K with $diam(K) < \delta$, such that the return map $f^p|_K$ is topologically conjugated to a model homeomorphism $h \in \mathcal{H} \cap RHom(D^m)$.*

Proof. We repeat the proof of the G_δ-set property of Lemma 4.8, putting $\mathcal{H} \cap RHom(D^m)$ instead of $\mathcal{H}$, and $\text{Hom}(M)$ instead of $C^0(M)$.

To show the denseness fix $f \in \text{Hom}(M)$ and $\epsilon > 0$. Let $\delta \in (0, \epsilon)$ be smaller the the ϵ-modulus of continuity of f and f^{-1}, and consider a periodic shrinking box K with $diam(K) < \delta$ (Lemma 4.5). Fix a homeomorphism $\phi : K \to D^m$. We will construct $g \in Hom(M)$ to be ϵ-near f in $\text{Hom}(M)$, with $g^p|_K =_\phi h \in \mathcal{H} \cap RHom(D^m)$.

From Definition 4.1 we know that the boxes $K, f(K), \ldots, f^{p-1}(K)$ are pairwise disjoint and that $f^p(K) \subset \text{int}(K)$. Denote $\overline{W} := f^{-1}(K)$. Since f is a homeomorphism, we deduce that $\overline{W}$ is a box as in Figure 4, such that $\overline{W} \cap f^j(K) = \emptyset$ for all $j = 0, 1, \ldots, p-2$ if $p \geq 2$, and $f^{p-1}(K) \subset W := \text{int}(\overline{W})$. Since $\text{diam}(K) < \delta$ we have $\text{diam}(\overline{W}) < \epsilon$.

Consider $\phi \circ f^p|_K \circ \phi^{-1} \in RHom(D^m)$. Applying part b) of Lemma 2.11, there exists a homeomorphism $\psi : D^m \to D^m$ such that

$$\psi|_{\partial D^m} = \text{id}|_{\partial D^m}, \qquad \psi \circ \phi \circ f^p|_K \circ \phi^{-1} = h \in \mathcal{H} \cap RHom(D^m).$$

So, we can construct $g \in \text{Hom}(M)$ such that $g(x) := f(x)$ for all $x \notin W$, and $g(x) := \phi^{-1} \circ \psi \circ \phi \circ f(x)$ for all $x \in W$. Since $\psi|_{\partial D^m}$ is the identity map, we obtain $g|_{\partial W} = f|_{\partial W}$. Thus, the above equalities define a continuous map $g : M \to M$. Besides g is invertible because $g|_W : W \to K$ is a composition of homeomorphisms, and $g|_{M \setminus W} = f|_{M \setminus W} : M \setminus W \to M \setminus K$ is also a homeomorphism. So, $g \in \text{Hom}(M)$. Moreover, by construction we have $|g(x) - f(x)| < \text{diam}(K) < \epsilon$ for all $x \in W$, and $g(x) = f(x)$ for all $x \notin W$. Also the inverse maps satisfy $|g^{-1}(x) - f^{-1}(x)| < \text{diam}(f^{-1}(K)) = \text{diam}(\overline{W}) < \epsilon$ for all $x \in K$, and $g^{-1}(x) = f^{-1}(x)$ for all $x \notin K$. Therefore $\|g - f\|_{Hom} < \epsilon$.

Finally, let us check that $g^p|_K$ is topologically conjugated to h:

$$g^p|_K = g|_{f^{p-1}(K)} \circ f^{p-1}|_K = g|_W \circ f^{p-1}|_K =$$
$$\phi^{-1} \circ \psi \circ \phi \circ f \circ f^{p-1}|_K =$$
$$\phi^{-1} \circ (\psi \circ \phi \circ f^p|_K \circ \phi^{-1}) \circ \phi = \phi^{-1} \circ h \circ \phi,$$

ending the proof of Lemma 4.9. □

Remark 4.10. In the proof of the dense property in Lemmas 4.8 and 4.9, once a periodic shrinking box K is constructed with period $p \geq 1$, we only need to perturb the map f inside $\overline{W} \bigcup \big(\bigcup_{j=0}^{p-1} f^j(K)\big)$, where $\overline{W} = f^{-1}(K)$ if f is a homeomorphism, and $\text{int}(\overline{W}) \supset f^{p-1}(W)$ otherwise. In both cases, by reducing $\delta > 0$ from the very beginning, if necessary, we can construct $\overline{W}$ such that $\text{diam}(\overline{W}) < \epsilon$ for a previously specified small $\epsilon > 0$.

Proof of Theorems 1 and 3. From Lemmas 4.8 and 4.9, a typical map $f \in C^0(M)$ and also a typical $f \in \text{Hom}(M)$ if $m = \dim(M) \geq 2$, has a periodic shrinking box K such that the return map $f^p|_K : K \to \text{int}(K)$ is conjugated to a model map $h \in \mathcal{H}$. We consider the homeomorphism $\phi : K \to D^m$ such that $\phi \circ f^p \circ \phi^{-1} = h \in \mathcal{H}$. Let $\psi := \phi^{-1}$. Besides, Lemma 3.1 states that every map $h \in \mathcal{H}$ has an h-invariant ergodic measure ν with infinite metric entropy for h. Consider the pull-back measure $\psi^*\nu$ defined by $(\psi^*\nu)(B) := \nu(\psi^{-1}(B \cap K)$ for all the Borel sets $B \subset M$. By construction, $\psi^*\nu$ is supported supported on $K \subset M$. Since ψ is a conjugation between h and $f^p|_K$, the pull-back measure $\psi^*\nu$ is f^p-invariant and ergodic for f^p, and besides $h_{\psi^*\nu}(f^p) = +\infty$.

From $\psi^*\nu$, we will construct an f-invariant and f-ergodic measure μ supported on $\bigcup_{j=0}^{p-1} f^j(K)$, with infinite metric entropy for f. Precisely, for each Borel set $B \subset M$, define

$$\mu(B) := \frac{1}{p} \sum_{j=0}^{p-1} (f^j)^*(\psi^*\nu)(B \cap f^j(K)), \tag{27}$$

where the pull back $(f^j)^*$ is defined by $(f^j)^*(\psi^*\nu)(A) := (\psi^*\nu)(f^{-j}(A))$ for any Borel set $A \subset M$. Applying Equality (27), and taking into account that $\psi^*\nu$ is f^p-invariant and f^p-ergodic, it is standard to check that μ is f-invariant and f-ergodic. Besides, from the convexity of the metric entropy function, we deduce that

$$h_\mu(f^p) = \frac{1}{p} \sum_{j=0}^{+\infty} h_{(f^j)^*(\psi^*\nu)}(f^p) = +\infty.$$

Finally, recalling that $h_\mu(f^p) \leq p\, h_\mu(f)$ for any f-invariant measure μ and any natural number $p \geq 1$, we conclude that $h_\mu(f) = +\infty$. $\square$

5. Good sequences of periodic shrinking boxes

The purpose of this section is to prove Theorems 2 and 4.

Definition 5.1. Let $f \in C^0(M)$ and let $K_1, K_2, \dots, K_n, \dots$ be a sequence of periodic shrinking boxes for f. We call $\{K_n\}_n$ *good* if it has the following properties (see Figure 5):

• $\{K_n\}_{n\geq 1}$ is composed of pairwise disjoint boxes.
• There exists a natural number $p \geq 1$, independent of n, such that K_n is a periodic shrinking box for f with a period p_n, a multiple of p.
• There exists a sequence $\{H_n\}_{n\geq 0}$ of periodic shrinking boxes, all with period p, such that $K_n \cup H_n \subset H_{n-1},\ K_n \cap H_n = \emptyset$ for all $n \geq 1$, and $\operatorname{diam}(H_n) \to 0$ as $n \to +\infty$.

Remark. Definition 5.1 implies that $\bigcap_{n\geq 1} H_n = \{x_0\}$, where x_0 is periodic with period p. Furthermore, for any $j \geq 0$ we have

$$d(f^j(K_n), f^j(x_0)) \leq \operatorname{diam}(f^j(H_{n-1})) \leq \max_{0\leq k\leq p-1} \operatorname{diam}(f^k(H_{n-1})) \overset{n\to\infty}{\to} 0,$$

and thus

$$\lim_{n\to+\infty} \sup_{j\geq 0} d(f^j(K_n), f^j(x_0)) = 0. \tag{28}$$

Lemma 5.2. *Typical maps $f \in C^0(M)$, and typical $f \in Hom(M)$ if $m \geq 2$, have a good sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes such that the return maps $f^{p_n}|_{K_n}$ are topologically conjugated to model maps.*

Proof. To see the G_δ property assume that f has a good sequence $\{K_n\}_n$ of periodic shrinking boxes. For each fixed n, the boxes K_n and H_n are also periodic shrinking with periods p_n and p respectively, for all g in an open set in $C^0(M)$ or in $\operatorname{Hom}(M)$ (see Definition 4.1). Taking the intersection of such open sets for all $n \geq 1$, we deduce that the same sequence $\{K_n\}$ is also a good sequence of periodic shrinking boxes for all g in a G_δ-set. Now, also assume that $f^{p_n}|_{K_n}$ is a model map for all $n \geq 1$. From Lemmas 4.8 and 4.9, for each fixed $n \geq 1$, the family of continuous maps g such that the return map $g^{p_n}|_{K_n}$ is topologically conjugated to a model, is a G_δ-set in $C^0(M)$ or in $\operatorname{Hom}(M)$. The (countable) intersection of all those G_δ-sets, produces a G_δ-set, as wanted.

To prove denseness fix $f \in C^0(M)$ or $f \in \operatorname{Hom}(M)$, and $\epsilon > 0$. We will construct g in the ϵ-neighborhood of f and a good sequence of periodic shrinking boxes K_n for g such that, $g^{p_n}|_{K_n} =_\phi h_n \in \mathcal{H}$ for all $n \geq 1$.

Typical maps $\widetilde{f} \in C^0(M)$ and typical $\widetilde{f} \in \operatorname{Hom}(M)$ have a periodic shrinking box H_0 with period $p \geq 1$, such that $\widetilde{f}^p|_{H_0} =_\phi h \in \mathcal{H}$ (Lemmas 4.8 and 4.9). Take such $\widetilde{f}$ in the $(\epsilon/6)$-neighborhood of f. Since $\widetilde{f}^p : H_0 \to \operatorname{int}(H_0) \subset H_0$ is continuous, by the Brouwer Fixed Point Theorem there exists a periodic point $x_0 \in int(H_0)$ of period p. Lemma 3.1 and the argument at the end of the proofs of Theorems 1 and 3, show that the map $\widetilde{f}$ has an ergodic measure μ supported on

$\bigcup_{j=0}^{p-1} \widetilde{f}^j(H_0)$ such that $h_\mu(\widetilde{f}) = +\infty$. Therefore, by Poincaré Recurrence Lemma, there exists some recurrent point $y_1 \in \text{int}(H_0)$ for $\widetilde{f}$ such that $y_1 \neq x_0$ (see Figure 5).

Choose $\delta_1 > 0$ small enough and construct a box $\overline{B}_1$ such that $y_1 \in B_1 := \text{int}(\overline{B}_1)$, $\text{diam}(\overline{B}_1) < \delta_1$, $x_0 \notin \overline{B}_1$ and $\overline{B}_1 \subset \text{int}(H_0)$. We repeat the proofs of the dense property of Lemmas 4.2 and 4.5, using the recurrent point y_1 instead of x_0, and the box $\overline{B}_1$ instead of $\overline{B}$ (see Figure 3). So, we deduce that there exists an $(\epsilon/6)$-perturbation f^* of $\widetilde{f}$, and a periodic shrinking box $K_1 \subset \overline{B}_1$ for f^*, with some period $p_1 \geq p$ (see Figure 5). Besides, f^* coincides with $\widetilde{f}$ in $H_0 \setminus B_1$ (recall Remark 4.3). Therefore, the same periodic point x_0 of $\widetilde{f}$ survives for f^*, and the same initial box H_0 is still periodic shrinking with period p for f^*. So, the compact sets of the family $\{{f^*}_1^j(H_0)\}_{j=0,1,\ldots,p-1}$ are pairwise disjoint, and $f^{*p}(H_0) \subset \text{int}(H_0)$. This implies that the period p_1 of the new periodic shrinking box K_1 for f^*, is a multiple of p.

Now, we apply the proofs of the dense property of Lemmas 4.8 and 4.9, using the shrinking box K_1 instead of K (see Figure 4). We deduce that there exists an $(\epsilon/6)$-perturbation g_1 of f^*, such that K_1 is still a periodic shrinking box for g_1 with the same period p_1, but besides, the return map is now $g_1^{p_1}|_{K_1} = h_1 \in \mathcal{H}$. Taking into account Remark 4.10, we can construct g_0 to coincide with f^* in the complement of $\overline{W}_1 \bigcup \left(\bigcup_{j=0}^{p_1-1} f^{*j}(K_1) \right)$, where $\overline{W}_1 \supset f^{p-1}(K_1)$ is a box, small enough not to contain the periodic point x_0, and to be contained in the interior of the shrinking box H_0. Therefore, x_0 and H_0 are still periodic with period p for g_0.

To summerize, we have built the periodic shrinking boxes H_0 and K_1 for a continuous map or homeomorphism g_1, with periods p and p_1 respectively, where p_1 is multiple of p, and a periodic point $x_0 \in \text{int}(H_0)$ (see Figure 5), such that:

$$K_1 \subset H_0 \setminus \{x_0\}, \quad g_1^{p_1}|_{K_1} =_\phi h_1 \in \mathcal{H} \quad \text{and}$$

$$\|g_1 - f\| < \|g_1 - f^*\| + \|f^* - \widetilde{f}\| + \|\widetilde{f} - f\| < \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2}.$$

We proceed by induction on $n \geq 1$, assume that $H_0, \ldots, H_{n-1}$ and $K_1, \ldots, K_n$ are periodic shrinking boxes (see Figure 5) of $g_n \in C^0(M)$ or $g_n \in \text{Hom}(M)$, with periods p and $p_1, \ldots, p_n$ respectively, where p_i is multiple of p, and that $x_{n-1} \in \text{int}(H_{n-1})$ is a periodic point of period p for g_n. Assume also that $K_n \subset H_{n-1} \setminus \{x_{n-1}\}$, that for $1 \leq j \leq n-1$

$$(29) \quad H_j, K_j \subset H_{j-1}; \; H_j \cap K_j = \emptyset; \; \text{diam}(H_j) < \frac{\epsilon}{2^j}; \; g_n^{p_j}|_{K_j} =_\phi h_j \in \mathcal{H},$$

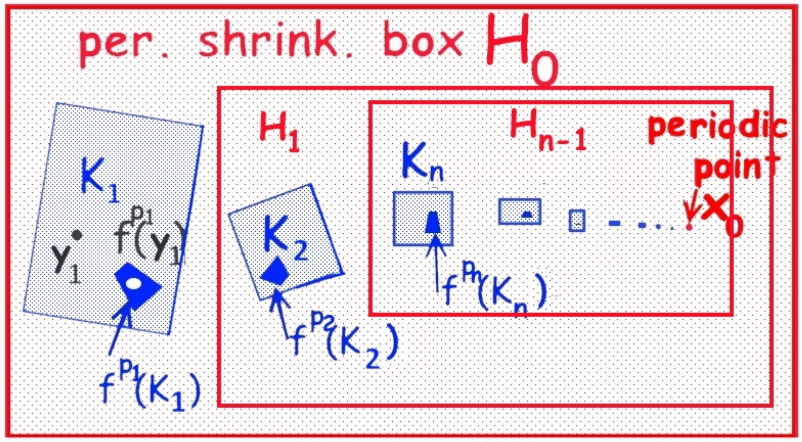

FIGURE 5. Construction of a good sequence of periodic shrinking boxes.

and that we have a finite number of the previously constructed maps $g_1, \ldots, g_n$ such that

$$\|g_1 - f\| < \frac{\epsilon}{2}, \quad \|g_j - g_{j-1}\| < \frac{\epsilon}{2^j} \quad \forall\, j = 2, \ldots, n. \tag{30}$$

We will construct g_{n+1} and the boxes H_n and K_{n+1} that satisfy the above properties for $n+1$ instead of n, and such that for all $j = 1, \ldots, n$, the boxes H_{j-1} and K_j are still periodic shrinking for g_{n+1} with the same periods p, p_j.

From the inductive hypothesis, g_n has a periodic shrinking box H_{n-1} of period p, a periodic point $x_{n-1} \in \operatorname{int}(H_{n-1})$ of period p, and a periodic shrinking box $K_n \subset H_{n-1} \setminus \{x_{n-1}\}$ of period p_n, a multiple of p. We choose $0 < \widetilde{\delta}_n < \epsilon/2^n$ small enough, and construct a box $\overline{\widetilde{B}_n} \subset H_{n-1}$ containing the periodic point x_{n-1} in its interior, disjoint from K_n, and such that $\operatorname{diam}(\overline{\widetilde{B}_n}) < \widetilde{\delta}_n$. Repeating the proof of the density properties in Lemmas 4.2 and 4.5 (putting x_{n-1} instead of x_0, and $\widetilde{\delta}_n > 0$ small enough), we construct an $\epsilon/(3 \cdot 2^{n+1})$-perturbation $\widetilde{g}_n$ of g_n and a periodic shrinking box $H_n \subset \widetilde{B}_n := \operatorname{int}(\overline{\widetilde{B}_n})$ for $\widetilde{g}_n$. Besides, since x_{n-1} is a periodic point with period p, the period of H_n can be made equal to p (see Remark 4.6). By construction $H_n \subset \widetilde{B}_n \subset H_{n-1}$ is disjoint with K_n and with ∂H_{n-1}. To construct $\widetilde{g}_n$ we only need to modify g_n inside $\widetilde{B}_n$ (recall Remark 4.3). Therefore the same periodic shrinking boxes $H_0, H_1, \ldots, H_{n-1}$ and $K_1, K_2, \ldots, K_n$ of g_n, are preserved for $\widetilde{g}_n$, with the same periods.

Now, as in the proof of Lemmas 4.8 and 4.9, we construct a new $\epsilon/(3 \cdot 2^{n+1})$-perturbation g_n^* of $\widetilde{g}_n$, such that we can ϕ-conjugate $g_n^{*p}|_{H_n}$ to a map in $\mathcal{H}$. To construct g_n^* we only need to modify $\widetilde{g}_n$ in $\widetilde{W}_n \cup (\bigcup_{j=0}^{p-1} g_n^j(H_n)$, where $\widetilde{W}_n$ is a neighborhood of $\widetilde{g}_n^{p-1}(H_n)$ that can be taken arbitrarily small (see Remark 4.10). Therefore we do not need to modify $\widetilde{g}_n$ or g_n outside H_{n-1} or inside K_n. We conclude that the same shrinking boxes $K_1, \ldots, K_n; H_0, \ldots, H_{n-1}$ for $\widetilde{g}_n$ and g_n, are still periodic shrinking for g_n^*, with the same periods and that $g_n^{*p_j}|_{K_j} = \widetilde{g}_n^{p_j}|_{K_j} =_\phi h_j$ for all $j = 1, \ldots, n$.

When modifying g_n inside $H_{n-1} \setminus K_n$ to obtain $\widetilde{g}_n$ and g_n^*, the periodic point x_{n-1} of period p for g_n, may not be preserved. But since $H_n \subset H_{n-1} \setminus K_n$ is a periodic shrinking box with period p for g_n^*, by the Brouwer Fixed Point Theorem, there exists a maybe new periodic point $x_n \in \operatorname{int}(H_n) \setminus K_n$ for g_n^*, with the same period p.

Since the return map $g_n^{*p}|_{H_n}$ is a model, there exists an ergodic measure with infinite entropy (see Lemma 3.1), supported on the g_n^*-orbit of H_n. Therefore, there exists a recurrent point $y_n \in \operatorname{int}(H_n)$ such that $y_n \neq x_n$. We choose $\delta_n > 0$ small enough, and a compact box $\overline{B}_n \subset \operatorname{int}(H_n) \setminus \{x_n\}$ such that $y_n \in B_n := \operatorname{int}(\overline{B}_n)$ and $\operatorname{diam}(\overline{B}_n) < \delta_n$. Repeating the above arguments, and if δ_n is small enough, we construct a new $\epsilon/(3 \cdot 2^{n+1})$-pertubartion g_{n+1} of g_n^* and a box $K_{n+1} \subset B_n$ that is periodic with some period p_{n+1} for g_{n+1}, and such that $g_{n+1}^{p_{n+1}}|_{K_{n+1}}$ is ϕ-conjugate to a map in $\mathcal{H}$.

To construct such a perturbation g_{n+1} of $\widetilde{g}_n$, we only need to modify $\widetilde{g}_n$ in B_n, and later in $W_{n+1} \cup (\cup_{j=0}^{p_{n+1}} \overline{g}_n(K_{n+1})$ (recall Remarks 4.3 and 4.10), where W_{n+1} is a small set (provided that $\delta_n > 0$ is small enough). Therefore, g_{n+1} can be constructed so the point x_n is still periodic with period p for g_{n+1}, and the same boxes $H_0, H_1, \ldots, H_n, K_1, \ldots, K_n$ are still shrinking periodic for g_{n+1}, with the same periods. Moreover, $g_{n+1}^{p_j}|_{K_j} = g_j^{p_j}|_{K_j} = h_j \in \mathcal{H}$ for all $j = 1, \ldots, K_n$.

In particular H_n is periodic shrinking with period p for g_{n+1}. And it contains K_{n+1} by construction. This implies that the period p_{n+1} of K_{n+1} is a multiple of p. By construction we have,

$$\|g_{n+1} - g_n\| \leq \|g_{n+1} - g_n^*\| + \|g_n^* - \widetilde{g}_n\| + \|\widetilde{g}_n - g_n\| < 3 \cdot \frac{\epsilon}{3 \cdot 2^{n+1}} = \frac{\epsilon}{2^{n+1}}.$$

Besides $\operatorname{diam}(H_n) < \widetilde{\delta}_n < \epsilon/2^n$. We have constructed g_{n+1} and the boxes H_n and K_{n+1} that satisfy the inductive properties for $n+1$ instead of n, as wanted.

We have constructed a sequence $\{g_n\}_{n\geq 1}$ of continuous maps or homeomorphisms on M, and two sequences $\{H_n\}_{n\geq 1}$, $\{K_n\}$ of compact boxes such that the Properties (29) and (30) are satisfied for all $n\geq 1$. Since $\|g_{n+1}-g_n\|\leq \epsilon/2^{n+1}$ for all $n\geq 1$ the sequence $\{g_n\}_{n\geq 1}$ is Cauchy. So, there exists a limit map g. And since g_n is an ϵ-perturbation of f for all $n\geq 1$, the limit map g also is. Finally, by construction we have $g_k(x)=g_n(x)$ for all $x\in\bigcup_{j=0}^{p_n} g_n^j(K_n)$, $g_k^{p_n}|_{K_n}=h_n\in\mathcal{H}$ for all $n\geq 1$ and for all $k\geq n$. So, we conclude that $\{K_n\}_{n\geq 1}$ is a good sequence of periodic shrinking boxes for g, and that $g^{p_n}|_{K_n}=_\phi h_n\in\mathcal{H}$, as wanted. □

Remark 5.3. As a consequence of Lemmas 5.2 and 3.1 (after applying the same arguments at the end of the proof of Theorems 1 and 3), typical continuous maps and homeomorphisms f have a sequence of ergodic measures μ_n, each one supported on the f-orbit of a box K_n of a good sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes for f, satisfying $h_{\mu_n}(f)=+\infty$ for all $n\geq 1$.

Let $\mathcal{M}$ denote the metrizable space of Borel probability measures on a compact metric space M, endowed with the weak* topology. Fix a metric dist^* in $\mathcal{M}$.

Lemma 5.4. *For all $\epsilon>0$ there exists $\delta>0$ that satisfies the following property: if $\mu,\nu\in\mathcal{M}$ and $\{\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_r\}$ is a finite family of pairwise disjoint compact balls $\overline{B}_i\subset M$, and if* $\mathrm{supp}(\mu)\cup\mathrm{supp}(\nu)\subset\bigcup_{i=1}^r\overline{B}_i$, and $\mu(\overline{B}_i)=\nu(\overline{B}_i)$, $\mathrm{diam}(\overline{B}_i)<\delta$ for all $i=1,2,\ldots,r$, *then* $\mathrm{dist}^*(\mu,\nu)<\epsilon$.

Proof. If $M=[0,1]$ the proof can be found for instance in [CT, Lemma 4]. If M is any other compact manifold of finite dimension $m\geq 1$, with or without boundary, just copy the proof of [CT, Lemma 4] by substituting the pairwise disjoint compact intervals $I_1,I_2,\ldots,I_r\subset[0,1]$ in that proof, by the family of paiwise disjoint compact boxes $\overline{B}_1,\overline{B}_2,\ldots,\overline{B}_r\subset M$. □

Proofs of Theorems 2 and 4. For any $\epsilon>0$, take $\delta>0$ as in Lemma 5.4. Applying Lemma 5.2, typical continuous maps or homeomorphisms f have a good sequence of periodic shrinking boxes $\{K_n\}_{n\geq 1}$, and a sequence $\{\mu_n\}$ of ergodic f-invariant measures such that $h_{\mu_n}(f)=+\infty$ (see Remark 5.3) and such that $\mathrm{supp}(\mu_n)\subset\bigcup_{j=1}^{p_n-1}f^j(K_n)$, where $p_n=h_n\cdot p$ is the period of the shrinking box K_n. Taking into account that $\{f^j(K_n)\}_{0\leq j\leq p_n-1}$ is a family of pairwise disjoint compact sets, and $f^{p_n}(K_n) \subset \mathrm{int}(K_n)$, we deduce

$$\mu(f^j(K_n)) = \mu(K_n) = \frac{1}{p_n} = \frac{1}{h_n\, p} \quad \forall\ j = 0, 1, \ldots, p_n,$$

(even if f were non invertible).

From Definition 5.1, there exists a periodic point x_0 of period p such that $\lim_{n\to+\infty} \sup_{j\geq 0} d(f^j(K_n, f^j(x_0)) = 0$. Therefore, there exists $n_0 \geq 1$ such that $d(f^j(K_n, f^j(x_0)) < \delta/2$ for all $j \geq 0$ and for all $n \geq n_0$. Consider the family of pairwise disjoint compact balls $\overline{B}_0$, $\overline{B}_1$, $\ldots$, $\overline{B}_{p-1}$, centered at the points $f^j(x_0)$ and with radius $\delta/2$, we obtain $f^j(K_n) \subset \overline{B}_{j(\mathrm{mod}.p)}$ for all $j \geq 0$ and for all $n \geq 0$. Therefore,

$$\mu_n(\overline{B}_j) = \frac{1}{p} \quad \forall\ j = 0, 1, \ldots, p-1, \qquad \forall\ n \geq n_0.$$

Finally, applying Lemma 5.4, we conclude $\mathrm{dist}^*(\mu_n, \mu_0) < \epsilon$ for all $n \geq n_0$, where $\mu_0 := (1/p)\sum_{j=0}^{p-1} \delta_{f^j(p)}$ is the f-invariant probability measure supported on the periodic orbit of x_0, which has zero entropy. $\square$

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