PREPRINT

Persistence of the Feigenbaum attractor in one-parameter families

Eleonora Catsigeras and Heber Enrich *

Preprint PREMAT 97/11 (1997). Published in Commun. Math. Phys. 207 (1999) pp. 621-640

Resumen

Consider a map ψ_0 of class C^r for large r of a manifold of dimension n greater or equal than 2 having a Feigenbaum attractor. We prove that any such ψ_0 is a point of a local codimension-one manifold of C^r transformations also exhibiting Feigenbaum attractors. In particular, the attractor persists when perturbing one-parameter family transversal to that manifold at ψ_0. We also construct such a transversal family for any given ψ_0, and apply this construction to prove a conjecture by J. Palis stating that a map exhibiting a Feigenbaum attractor can be perturbed to obtain homoclinic tangencies.

1 Introduction

The existence of a local codimension-one manifold of transformations exhibiting Feigenbaum attractors is well known from the Feigenbaum- Coullet-Tresser theory [CT 1978, Fe 1978, Fe 1979] (for proofs see [La 1982, Su 1991, Ly 1999]) for unimodal real analytic maps in the interval that are in a neighborhood of the fixed map of the doubling renormalization. It is a consequence of the hyperbolicity of this fixed point, as proved by Lanford III [La 1982]. It is also true for n-dimensional transformations [CEK 1981].

The existence of such a local codimension-one manifold was recently proved to be valid not only for real analytic transformations but also in the $C^{2+\epsilon}$ topology, as shown in [Da 1996] for maps in the interval (see also [Su 1991]), and in the C^r topology, for sufficiently large r, for maps in $n \geq 2$ dimensions [CE 1998].

All these results are proved to be true only in a small neighborhood of the fixed map for $n \geq 2$ (for global results in dimension 1, see [Su 1991], [Ly 1999]). As the renormalization is not invertible neither differentiable, we cannot deduce that there is a global codimension-one manifold, in the space of n-dimensional transformations with $n \geq 2$ containing Feigenbaum attractors.

We address here the question of whether there can exist, far away from the fixed map, a whole open set of transformations exhibiting Feigenbaum attractors, or if, on the contrary, these attractors can always be destroyed by arbitrarily small perturbations. In dimension one this is a difficult question whose solution requires elaborate complex analytic methods (see [Su 1991], [Ly 1999], [Mc 1994]). But, in dimension two or greater where the problem was open so far, we will show here, using explicit constructions, that the Feigenbaum attractor is an unstable phenomenon in the space of C^r maps. We describe its unstability: although unstable in the space of C^r transformations, the Feigenbaum attractor is persistent in the space of one-parameter families of such transformations. In other words, it is a codimension-one phenomenon.

*Both authors: IMERL Fac. Ingenieria Uruguay. Address: IMERL C.C. 30 Montevideo Uruguay. E-mail: eleonora@fing.edu.uy

The persistence of the Feigenbaum attractor in one-parameter families, when combined with some previously known results, has some immediate consequences. For instance, a one parameter family of n-dimensional transformations, near the quadratic family, will always have a Feigenbaum attractor. These families appear when unfolding generically a homoclinic tangency [PT 1993]. As a consequence, a theorem of Colli [Co 1996] can be applied to prove that, near such unfolding, one can find transformations exhibiting infinitely many coexisting Feigenbaum attractors.

To describe the unstability of the Feigenbaum attractor we construct a one-parameter family passing through the given map, showing at one side a sequence of period doubling bifurcations, and at the other side, a sequence of homoclinic bifurcations. The construction of such a family exploits the existence of a good spatial direction, along which the perturbation can be made. This direction is not available when working with unimodal maps of the interval, due to the existence of critical points. That is the reason why the arguments work only in dimension greater than or equal to 2.

The one-parameter families constructed in this work are of the kind described in [GST 1989] to provide examples of Feigenbaum attractors for smooth embeddings of the 2-disk. We show that any Feigenbaum attractor is a point of such a family.

The construction of a good one-parameter family provides a proof of a conjecture of J. Palis: the map having a Feigenbaum attractor can be perturbed to exhibit a homoclinic tangency. Thus, this attractor is near the chaotic phenomena that appear when unfolding a homoclinic tangency (as strange attractors for instance).

We thank Jacob Palis, Welington de Melo, Charles Tresser, Eduardo Colli, Raúl Ures and Miguel Paternain for their suggestions. We are very thankfull to the referee for his corrections.

Statement of the main theorems and their corollaries.

To state the theorems in the C^r topology, we fix r sufficiently large. ($r \geq 8$ is sufficient to prove the theorems: we do not seek for an optimum value of r). The space of transformations $C^r(\mathcal{D})$ is the open set of the Banach space of C^r maps from a n-dimensional closed ball $\mathcal{D}$ of R^n to R^n ($n \geq 2$), such that the image of $\mathcal{D}$ is contained in the interior of $\mathcal{D}$. In section 2 we define the doubling renormalization $\mathcal{R}$ of certain maps in $C^r(\mathcal{D})$. We then define what we call a Feigenbaum attractor in dimension $n \geq 2$. We call it so because of its geometric similarity to the Feigenbaum attractors of unimodal maps in the interval. But it should be remarked that contributions to the theory in dimension greater than one, and also in dimension one, is now a day far from the origin of the name of such attractors. In dimension two or greater most known results are based on the renormalization theory of unimodal maps of the interval. For details of one-dimensional renormalization theory see [MS 1993].

Roughly speaking a Feigenbaum attractor is a Cantor set attractor of an infinitely doubling renormalizable map, whose dynamics microscopically converges, in the C^r topology, to that of the real analytic map that is a fixed point of the renormalization operator.

In section 3 we define an appropriate topology in the space $\mathcal{F}$ of one-parameter families of maps in $C^r(\mathcal{D})$, and we define the persistence of the Feigenbaum attractor in one-parameter families: each X in $\mathcal{F}$, near a given family, passes through a map exhibiting such an attractor.

In section 4 we prove the following

Theorem 1 *If $\Psi = \{\psi_a\}_{a \in I} \in \mathcal{F}$ is such that ψ_0 has a Feigenbaum attractor K and after some finite number N of doubling renormalizations the family $\tilde{\Psi} = \{\mathcal{R}^N \psi_a\}_{a \in (-\epsilon, \epsilon)}$, for some small $\epsilon > 0$, intersects transversally the local stable manifold of the fixed map of the renormalization, then K is persistent in one-parameter families near Ψ and there exists a local codimension-one* *differential manifold $\mathcal{M}$ in the space $C^r(\mathcal{D})$, transversally intersecting Ψ at ψ_0, formed of maps exhibiting Feigenbaum attractors.*

One may be particularly interested in taking Ψ as a family of C^r diffeomorhisms, but the last theorem works also for non injective n-dimensional maps.

As shown in [La 1982] the quadratic family, after being renormalized a finite number of times, intersects transversally the stable manifold of the fixed map of the renormalization. This is true for unimodal maps in the interval. It can be easily generalized to maps in n dimensions, where the quadratic family acquires the form:

$$\psi_a(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_n, 0, \ldots, 0, 1 - a x_n^2)$$

Thus we obtain the following:

Corollary 1 *The Feigenbaum attractor exhibited for $a = 1.401155\ldots$ in the quadratic family of dimension $n \geq 2$ is persistent in one-parameter families. Moreover, there exists in the space of C^r transformations a local codimension-one manifold $\mathcal{M}$, containing $\psi_{1.401155\ldots}$, of maps exhibiting Feigenbaum attractors, and $\mathcal{M}$ is transversal in $C^r(\mathcal{D})$ to the quadratic family.*

Some properties of the dynamics near homoclinic tangencies are deduced from the last corollary and from the theorem of Colli [Co 1996]:

Corollary 2 (Colli) *Let $h_0 \in \mathit{Diff}^{\,\infty}(M)$, where M is a 2-dimensional manifold, be such that h_0 has a homoclinic tangency between the stable and unstable manifolds of a dissipative hyperbolic saddle. Then there exists an open set $\mathcal{V} \subset \mathit{Diff}^{\,\infty}(M)$ such that*

- *$h_0 \in \overline{\mathcal{V}}$*
- *there exists a dense subset $\mathcal{S} \subset \mathcal{V}$ such that for all $h \in \mathcal{S}$, h exhibits infinitely many coexisting Feigenbaum attractors.*

In [YA 1983] it is proved that a one-parameter family of transformations forming a horseshoe must pass through a sequence of period doubling bifurcations. This implies the appearance of cascades of period doubling bifurcations when unfolding a generic homoclinic tangency. The map where these bifurcations accumulate has a Cantor set attractor. This raises two questions:

Is this Cantor set a Feigenbaum attractor? (i.e. microscopically looks like the Cantor set attractor of the fixed map of the renormalization).

Is the sequence of period doubling bifurcations *pure,* in the sense that, for all sufficiently large N, the period 2^N orbit that is born at each bifurcation does not suffer other bifurcations before the next period doubling bifurcation appears?

As shown in [PT 1993] a family generically unfolding a quadratic homoclinic tangency can be properly renormalized so that it is arbitrarily close to the quadratic family. Therefore, from Corollary 1 we can deduce the following:

Corollary 3 *Let the family $\{h_\mu\}_\mu$, $h_\mu \in \mathit{Diff}^{\,\infty}(M)$, where M is a 2-dimensional manifold, generically unfold a quadratic homoclinic tangency for $\mu = 0$ between the stable and unstable manifolds of a dissipative hyperbolic saddle.*

Then, for μ sufficiently close to 0, this family passes through a pure sequence of period doubling bifurcations, and the Cantor set attractor, exhibited for the parameter value where this sequence accumulates, is a Feigenbaum attractor.

Theorem 1 asserts that the Feigenbaum attractor persists for families near a given family Ψ, which verifies, after being renormalized, a certain transversality condition. For any transformation ψ_0 having a Feigenbaum attractor, it is natural to raise the general question of whether ψ_0 always belongs a local codimension-one manifold of transformations with Feigenbaum attractors. In order to apply Theorem 1, one needs to prove the existence of a good family Ψ passing through ψ_0. However, since the renormalization is not surjective, the construction of such a family is not trivial. This question is treated in section 5, where we prove:

Theorem 2 *If $\psi_0 \in C^r(\mathcal{D})$ has a Feigenbaum attractor, then there exists a one-parameter family $\Psi \in \mathcal{F}$, passing through ψ_0, verifying the hypothesis of Theorem 1, and there exists a local codimension-one manifold $\mathcal{M}$ in the space of C^r maps, passing through ψ_0, transversal to the family Ψ, such that all $\chi \in \mathcal{M}$ has a Feigenbaum attractor.*

As a consequence of Theorem 2 and of a theorem in [CE 1998] we obtain the following:

Corollary 4 *If $\psi_0 \in C^r(\mathcal{D})$ has a Feigenbaum attractor then there exists a one-parameter family $\Psi = \{\psi_t\}_{t\in(-\epsilon,\epsilon)}$ such that for $t < 0$ passes through a pure sequence of period doubling bifurcations, accumulating at ψ_0, and for a sequence of positive parameter values t_m, converging to 0, ψ_{t_m} exhibits a homoclinic tangency.*

This last corollary generalizes previous theorems of approximation with homoclinic tangencies of cascades of period doubling bifurcations, [Ca 1996, CE 1998], and is a contribution to a conjecture of J. Palis according to which global unstable phenomena can be perturbed to obtain homoclinic bifurcating maps [PT 1993].

2 The Feigenbaum attractor in n dimensions

Let $f_0 : [-1,1] \mapsto [-1,1]$ be the *fixed map in the interval, i.e.*, the unique real analytic unimodal map in the interval such that $f_0(0) = 1,\ f_0''(0) < 0$ and

$$f_0(1)^{-1} \cdot f_0 \circ f_0(f_0(1) \cdot x) = f_0(x) \quad \forall x \in [-1,1]$$

The existence and unicity of f_0 was the central conjecture of the Feigenbaum-Coullet-Tresser theory [CT 1978, Fe 1978, Fe 1979] and was proved in [La 1982, Su 1991]. We denote λ to the number $-f_0(1) = 0.3995\ldots$. The map f_0 has a single fixed point in $[-1,1]$, which is larger than λ. The analytic map f_0 is symmetric: $f_0(x) = g_0(x^2)$ where g_0 is an analytic diffeomorphism from $[0,1]$ to $[-\lambda,1]$. It can be uniquely extended to an open interval.

There exists one single periodic orbit of f_0 of period 2^N for each natural $N \geq 0$, and this orbit is a hyperbolic repellor. The orbit of countably many points eventually fall on one of these repellors. All the other orbits of f_0 are attracted to a Cantor set K in the interval which we call *the (standard) Feigenbaum attractor in the interval.*

Definition 2.1 Let the dimension n be a natural number greater or equal than 2. Let

$$\phi_0(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_n, 0, \ldots, 0, f_0(x_n)) = (x_n, 0, \ldots, 0, g_0(x_n^2))$$

defined in a small compact neighborhood $\mathcal{D}$ in R^n of $[-\lambda,1] \times \{0\}^{n-2} \times [-\lambda,1]$ to its interior. It will be called the *fixed map in n dimensions.* It inherits the Cantor set attractor of the map f_0, that we call the *(standard) Feigenbaum attractor in n dimensions.*

Remark 2.2 Observe that the fixed map in n dimensions has a one dimensional character: it is an endomorphism of $\mathcal{D}$ endowing it to a one-dimensional image, contained in its interior, and following the graph of f_0.

The repellors of f_0 are transformed into periodic hyperbolic saddles of ϕ_0 with infinite contraction along their stable manifolds. There exist such a periodic orbit with period 2^N for each natural $N \geq 0$. The unstable manifolds of the saddles have dimension one and are contained in $\phi_0(\mathcal{D})$. The stable manifold of each saddle is the union of their preimages by ϕ_0, formed by horizontal $(n-1)$-dimensional hyperplanes intersected with $\mathcal{D}$.

All the orbits of ϕ_0, except those in the stable manifolds of the saddles, are attracted to the Feigenbaum attractor.

We note that the fixed point of f_0 has no other preimages in the interval $[-\lambda, 1]$. As a consequence, the stable manifold of the fixed point x_0 of ϕ_0 is $\phi_0^{-1}(x_0)$, which does not intersect $\phi_0(\mathcal{D})$ except at x_0.

We are interested in studying some Cantor set attractors for other n-dimensional maps, particularly for diffeomorphisms that might be far away from the fixed map. First, we generalize the definition of a *Feigenbaum attractor*: roughly speaking we want to include any Cantor set attracting all the orbits in a neighborhood except those in the stable manifolds of a countable set of periodic hyperbolic orbits, and that, up to a bounded deformation that microscopically approaches the identity, is the standard Feigenbaum attractor. To formalize the idea we need the following:

Definition 2.3 A *n-disk $\mathcal{D}$,* (or simply a *disk*), is the image by a C^r diffeomorphism of the unit closed ball of R^n. ($n \geq 2$).

In particular, the domain $\mathcal{D}$ of ϕ_0 can be chosen to be a n-disk.

To state our theorems we will work with fixed r that is greater or equal to a minimal specific value needed for each theorem. We did not look for an optimum value of r, but, as will be explained later $r \geq 8$ is good enough for both Theorems 1 and 2.

Definition 2.4 Given a n-disk $\mathcal{D}$, the space $C^r(\mathcal{D})$ is the open set of all the maps of C^r class from $\mathcal{D}$ to its interior, with the topology given by the C^r norm $\| \cdot \|_r$.

In some parts of this paper we will need to work with the whole Banach space of C^r maps from $\mathcal{D}$ to R^n although their images are not contained in the interior of $\mathcal{D}$. We will still denote it as $C^r(\mathcal{D})$, if there were no risk of confusion.

Definition 2.5 A map $\psi \in C^r(\mathcal{D})$ is *doubling renormalizable* if there exists a disk $\mathcal{D}_1 \subset \text{int}\mathcal{D}$ such that:

$$\psi(\mathcal{D}_1) \cap \mathcal{D}_1 = \emptyset$$

$$\psi^2(\mathcal{D}_1) \subset \text{int}(\mathcal{D}_1)$$

If ψ is doubling renormalizable and $\xi : \mathcal{D} \mapsto \mathcal{D}_1$ is a C^r diffeomorphism (called *change of variables*), the map $\mathcal{R}\psi$ defined as $\mathcal{R}\psi = \xi^{-1} \circ \psi \circ \psi \circ \xi$ is a *renormalized map* of ψ.

Note that doubling renormalizability is an open condition in $C^r(\mathcal{D})$. Also note that $\mathcal{R}\psi$ is not uniquely defined: small perturbations of the change of variables ξ give other renormalized map of ψ. When referring to the properties of $\mathcal{R}\psi$ we understand that there exists some renormalized map of ψ having these properties.

By induction is defined:

Definition 2.6 A map $\psi \in C^r(\mathcal{D})$ is *m-times (doubling) renormalizable* if it is $m-1$-times (doubling) renormalizable and its $m-1$-renormalized $\mathcal{R}^{m-1}\psi$ is doubling renormalizable. It is defined a *m-renormalized map of ψ* as $\mathcal{R}^m\psi = \mathcal{R}\mathcal{R}^{m-1}\psi$

Definition 2.7 A map $\psi \in C^r(\mathcal{D})$ is *infinitely (doubling) renormalizable* if it is m-times (doubling) renormalizable for all natural m.

Remark 2.8 The main example of infinite doubling renormalizable map is the fixed map ϕ_0, defined in 2.1. In fact, ϕ_0 is fixed by the renormalization $\mathcal{R}\phi_0 = \Lambda^{-1} \circ \phi_0 \circ \phi_0 \circ \Lambda = \phi_0$ where

$$\Lambda(x_1, x_2, \ldots, x_n) = (-\lambda x_1, \lambda x_2, \ldots \lambda x_{n-1}, g_0(\lambda^2 g_0^{-1}(x_n)))$$

being $\lambda = 0.3995\ldots$ To verify the identity $\mathcal{R}\phi_0 = \phi_0$ we used that $-\lambda f_0(x) = f_0 \circ f_0(\lambda x)$, which implies $-\lambda g_0(u) = g_0([g_0(\lambda^2 u)]^2)$ and so

$$\phi_0 \circ \Lambda(x_1, x_2, \ldots . x_n) = (g_0(\lambda^2 g_0^{-1}(x_n)), 0, \ldots, 0, -\lambda x_n)$$

Proposition 2.9 *The change of variables Λ and $\phi_0 \circ \Lambda$ are contractions.*

Proof: We must show that the derivatives $D\Lambda(x)$ and $D(\phi_0 \circ \Lambda)(x)$ have norm smaller than 1 for all $x \in \mathcal{D}$.

Looking at the expressions of Λ and $\phi_0 \circ \Lambda$, made explicit in the remark 2.8, and as $0 < \lambda < 1$, it is enough to show that

$$\left| \frac{\lambda^2 g_0'(\lambda^2 g_0^{-1}(x))}{g_0'(g_0^{-1}(x))} \right| < 1$$

for all $x \in [-\lambda, 1]$. The above inequality is trivial if $g_0^{-1}(x) = 0$. Now, if $g_0^{-1}(x) = u^2 \neq 0$, using that $2u g_0'(u^2) = f_0'(u)$, we obtain

$$\left| \frac{\lambda^2 g_0'(\lambda^2 g_0^{-1}(x))}{g_0'(g_0^{-1}(x))} \right| = \left| \frac{\lambda f_0'(\lambda u)}{f_0'(u)} \right| < \lambda < 1$$

because $f_0'' < 0$ and $0 < \lambda u < u$.

■

Observe that we have indeed proved that $\|D\Lambda(x)\| = \|D(\phi_0 \circ \Lambda)(x)\| = \lambda = 0.3995\ldots < 1$.

Being ϕ_0 fixed by the renormalization, any map χ in a neighborhood $\mathcal{U}$ of ϕ_0 in $C^r(\mathcal{D})$ is doubling renormalizable.

From the Feigenbaum-Coullet-Tresser theory we have the hyperbolicity of ϕ_0. In [CE 1998] it is shown that the change of variables $\xi(\chi)$ can be chosen depending continuously on $\chi \in \mathcal{U}$, such that $\xi(\phi_0) = \Lambda$ and such that the renormalization $\mathcal{R}$, now *uniquely defined* as $\mathcal{R}\chi = \xi(\chi)^{-1} \circ \chi \circ \chi \circ \xi(\chi)$, has a hyperbolic behavior in $\mathcal{U}$. The renormalization $\mathcal{R}$ is not Fréchet differentiable in $C^r(\mathcal{D})$, so we can not expect ϕ_0 to be a hyperbolic fixed point in the differentiable sense. But we can define the local stable set of ϕ_0 as

$$W^s(\phi_0) = \{\chi \in C^r(\mathcal{D}) : \chi \text{ is infinitely doubling renormalizable and } \mathcal{R}^m\chi \in \mathcal{U} \text{ for all } m \geq 0\},$$

and the local unstable set of ϕ_0 as

$$W^u(\phi_0) = \{\chi \in C^r(\mathcal{D}) : \text{ for all } m \geq 0 \text{ there exists } \chi_m \in \mathcal{U} \text{ such that } \mathcal{R}^m\chi_m = \chi\}$$

Eventually reducing the neighborhood $\mathcal{U}$ of ϕ_0 in $C^r(\mathcal{D})$, we have the following result:

Theorem 2.10 *Let $r \geq 6$. In the functional space $C^r(\mathcal{D})$ of C^r maps of the n-dimensional disk $\mathcal{D}$, let ϕ_0 be the fixed map of the renormalization. Then $W^s(\phi_0)$ is a local codimension-one C^1 manifold, $W^u(\phi_0)$ is a one-parameter differentiable family $\{\phi_t\}_t$, and both intersect transversally at ϕ_0. Moreover, the renormalized maps of any map in $W^s(\phi_0)$ converge to ϕ_0.*

Note on the proof: The last theorem is proved in [CE 1998](Theorem 3.6) for r large enough. It can be shown that $r \geq 6$ is sufficient, using some adjustments to the numerical computation of the derivatives of the fixed map f_0 in the interval: in the lemma 2.4 of [CE 1998] one can found an inequality to fix an integer number h

$$\lambda^h(a^h\lambda^{-2} + b) < 1$$

where $\lambda = 0.3995\ldots$, $b = 1/\lambda^2$ and $a = |f_0'(\lambda)|$. To prove the theorem 3.6 of [CE 1998] one uses $r \geq h + 2$.

Replacing the constant a by 1.19236, which is an upper estimation of its value, one obtains $h \geq 4$ and thus $r \geq 6$.

The estimation of a was computed using the first ten coefficients of $f_0(x)$ as a series of powers of x^2. These coefficients are explicitly determined in [La 1982].

Remark 2.11 As the last theorem is local in a neighborhood of the Feigenbaum map ϕ_0 in n dimensions, it is true also for any other n-dimensional map $\overline{\phi_0}$, conjugated to ϕ_0 by a C^{r+1} conjugation ξ between n-disks: in fact, the conjugation ξ defines a bijection of conjugated maps, that is a diffeomorphism from a neighborhood of ϕ_0 in $C^r(\mathcal{D})$ to a neighborhood of $\overline{\phi_0}$ in $C^r(\xi(\mathcal{D}))$, preserving the doubling renormalization.

To prove Theorem 2.10, ([CE 1998] Theorem 3.6) we used a fixed map φ_0 in n dimensions that does not coincide with the fixed map ϕ_0 defined in this work, but both are equivalent up to a conjugation. In fact: in [CE 1998], as in the paper of Collet, Eckmann and Koch [CEK 1981], the unimodal symmetric map of the interval $x \mapsto f_0(x) = g_0(x^2)$, with g_0 a real analytic diffeomorphism, gives place to the n-dimensional map φ_0 in a n-disk defined as

$$\varphi_0(x_1, x_2, \ldots, x_n) = (g_0(x_1^2 - \alpha x_n), 0, \ldots, 0)$$

where $\alpha \neq 0$ is constant.

In this work we are associating to the map f_0 in the interval, the n-dimensional map ϕ_0 in a disk defined as

$$\phi_0(x_1, x_2, \ldots, x_n) = (x_n, 0, \ldots, 0, f_0(x_n)).$$

Both transformations φ_0 and ϕ_0, defined in appropriate disks of R^n are equivalent up to a smooth conjugation. In fact, it can be verified that $\xi \circ \varphi_0 \circ \xi^{-1} = \phi_0$, where

$$\xi(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_1, x_2, \ldots, x_{n-1}, g_0(x_1^2 - \alpha x_n))$$

Here we have preferred the transformation ϕ_0 instead of φ_0 because the corollaries of the theorems become immediate.

The following theorem allows us to define the Feigenbaum attractors for maps in $C^r(\mathcal{D})$ far away from the fixed map:

Theorem 2.12 *Let $r \geq 1$. If $\psi \in C^r(\mathcal{D})$ is infinitely doubling renormalizable and, for some $m \geq 0$, $\mathcal{R}^m\psi \in W^s(\phi_0)$, then there exist a minimal Cantor set K such that $\psi(K) = K$, a neighborhood U of K and, for each sufficiently large N, a single periodic orbit of period 2^N in U, which is hyperbolic of saddle type. Moreover, K attracts all the orbits in U except those in the stable manifolds of these periodic orbits and there are no other periodic orbits in U. All the orbits of K are quasi-periodic.*

Proof:

ϕ_0 is infinitely renormalizable and $\mathcal{R}\phi_0 = \phi_0 = \Lambda^{-1} \circ \phi_0 \circ \phi_0 \circ \Lambda$, where the change of variables Λ is defined in the remark 2.8. By the proposition 2.9, Λ and $\phi_0 \circ \Lambda$ are contractions.

Any map χ in a neighborhood $\mathcal{U}$ of ϕ_0 is doubling renormalizable and $\mathcal{R}\chi = \xi^{-1} \circ \chi \circ \chi \circ \xi$, where $\xi = \xi(\chi)$, depending continuously on χ, is near Λ and $\chi \circ \xi$ is near $\phi_0 \circ \Lambda$. Therefore both ξ and $\chi \circ \xi$ are contractions with contraction rate bounded above by a constant $\beta < 1$, which is uniform for all $\chi \in \mathcal{U}$.

If besides χ belongs to $W^s(\phi_0)$, then it is infinitely renormalizable in $\mathcal{U}$, and its renormalized map $\mathcal{R}^m\chi$ is $\xi_m^{-1} \circ \chi^{2^m} \circ \xi_m$, where $\xi_m = \xi(\chi) \circ \xi(\mathcal{R}\chi) \circ \ldots \circ \xi(\mathcal{R}^{m-1}\chi)$.

As $\chi^j \circ \xi_m$, for $j = 0, 1, \ldots 2^m - 1$, can be written as the composition of m maps, each being either $\xi(\mathcal{R}^i\chi)$ or $\mathcal{R}^i\chi \circ \xi(\mathcal{R}^i\chi)$ for $i = 0, 1, \ldots, m-1$, then $\chi^j \circ \xi_m$ is a contraction with contraction rate bounded above by β^m.

Let us define

$$A_{j,m} = \chi^j \circ \xi_m(\mathcal{D})$$

for $m \geq 1$ and $0 \leq j \leq 2^m - 1$. It is a compact set with diameter smaller than $\beta^m \mathrm{diam}\mathcal{D}$.

We assert that, for fixed m, the 2^m sets $A_{j,m}$ are pair wise disjoint in j, in spite of χ be not necessarily injective. In fact $A_{0,1}$ and $A_{1,1}$ are disjoint because χ is doubling renormalizable. For the same reason, but using $\mathcal{R}\chi$ instead of χ, the sets $A_{0,2}$ and $A_{2,2}$ are disjoint contained in $A_{0,1}$. On the other hand $A_{1,2}$ and $A_{3,2}$ are both contained in $A_{1,1}$ and are transformed by χ in disjoint sets, so are disjoint. By induction the assertion is proved.

Let

$$K = \cap_{m=1}^{\infty} \cup_{j=0}^{2^m-1} A_{j,m}$$

The diameter of the disjoint compact sets $A_{j,m}$ for $j = 0, 1, \ldots$, converge to 0 as $m \to \infty$, and so K is a Cantor set.

It is easy to show that $\chi(K) \subset K$. In fact, $\chi(A_{j,m}) = A_{j+1,m}$ for $0 \leq j \leq 2^m - 2$ and $\chi(A_{2^m-1,m}) \subset A_{0,m}$.

As the diameter of $A_{j,m}$ converges to 0 with m, each point of K is quasi-periodic, but never periodic. So K is minimal and $\chi(K) = K$.

Now let us prove that K is an attractor for χ. To do this we will use the properties of the fixed map ϕ_0 remarked in 2.2: ϕ_0 has a single fixed point x_0 that is hyperbolic of saddle type. Its stable manifold $W^s(x_0)$ is the compact set $\phi_0^{-1}(x_0)$. All the forward orbits of ϕ_0, except those in $W^s(x_0)$, eventually enter the disk $\Lambda(\mathcal{D})$.

As $\Lambda^{-1} \circ \phi_0 \circ \phi_0 \circ \Lambda = \phi_0$ and $\phi_0(\mathcal{D}) \subset \mathrm{int}\ (\mathcal{D})$, we can choose an open set W such that

$$\phi_0^2 \circ \Lambda(\mathcal{D}) \subset W \subset \overline{W} \subset \mathrm{int}\ \Lambda(\mathcal{D})$$

So, once a orbit enters $\Lambda(\mathcal{D})$, its following iterates never escape the open set $\phi_0^{-1}(W) \cup W$.

Let us choose a small neighborhood H of x_0, disjoint with $\phi_0^{-1}(W) \cup W$, such that for ϕ_0, and also for all maps χ near ϕ_0 in $C^r(\mathcal{D})$, a forward orbit is in the stable manifold of the fixed point if it is totally contained in H.

As $\phi_0^{-1}(x_0) = W^s(x_0)$ we have that $\phi_0^{-1}(H) \supset W^s(x_0)$. Let us choose a open set V such that

$$\phi_0^{-1}(H) \supset \overline{V} \supset V \supset W^s(x_0)$$

As $\mathcal{D} - V$ is compact there exists a natural number N such that, for all $y \in \mathcal{D} - V$, either $\phi_0^N(y)$ or $\phi_0^{N+1}(y)$ belongs to W. In fact, all the forward orbits, except those in $W^s(\phi_0)$, eventually remain in $W \cup \phi_0^{-1}(W)$.

Now let us perturb ϕ_0 taking a map $\chi \in C^r(\mathcal{D})$ sufficiently near ϕ_0 so that:

- χ has a fixed hyperbolic saddle point $\overline{x}_0 \in H$.
- $\chi^j(y) \in H$ for all $j \geq 0$ implies $y \in W^s(\overline{x}_0)$.
- $\chi(\overline{V}) \subset H$
- $\chi^2 \circ \xi(\chi)(\mathcal{D}) \subset W \subset \overline{W} \subset \xi(\chi)(\mathcal{D})$.
- If $y \in \mathcal{D} - V$, then either $\chi^N(y)$ or $\chi^{N+1}(y)$ belongs to W.

We assert that all the forward orbits by χ, except those in $W^s(\overline{x}_0)$, eventually enter W, and so, into the disk $\xi(\chi)(\mathcal{D})$. In fact, if $y \in \mathcal{D} - V$ either $\chi^N(y)$ or $\chi^{N+1}(y)$ belongs to W. If $\chi^j(y) \in V$ for all $j \geq 0$, then $\chi^{j+1}(y) \in H$ for all $j \geq 0$ and so $y \in W^s(\overline{x}_0)$, proving our assertion.

Now we are ready to prove the proposition: if $\mathcal{R}^m\psi \to \phi_0$, the m-renormalized of ψ, for $m \geq M$, are all in $\mathcal{U}$ and sufficiently near ϕ_0 to verify the conditions described above. Thus for $\chi = \mathcal{R}^M(\psi)$ the invariant Cantor set K attracts all the orbits except those in the stable manifolds of the hyperbolic saddles of period 2^N of χ, for $N \geq 0$, that is of period 2^{N+M} of ψ.

As $\mathcal{R}^M(\psi) = \xi_M^{-1} \circ \psi^{2^M} \circ \xi_M$, where ξ_M is a diffeomorphism between $\mathcal{D}$ and a sub-disk $\mathcal{D}_M$, we obtain the thesis taking $U = \cup_{j=0}^{2M-1} \psi^{-j}\text{int}(\mathcal{D}_M)$.

$\blacksquare$

Note: From the proof of the theorem 2.12 observe that the Cantor set attractor K has bounded geometry in the sense that the diameter of the atoms $A_{j,m}$ decrease with a rate bounded below 1. Even more, when looking microscopically the decreasing rate tends to the number $\lambda = 0.3995\ldots$, that is a spatial universal constant defined for the fixed map ϕ_0. In fact, λ is the contraction rate of the change of variables in the proof of the proposition 2.9.

In [GT 1992] an example is given of a n dimensional infinitely renormalizable map whose renormalized maps do not converge to the fixed map ϕ_0. In spite of that, this example has a Cantor set attractor that verifies the thesis of the theorem 2.12. Its geometry is also bounded, but the bounds are different from λ.

Based in the theorem 2.12, and with the aim to extend the Feigenbaum-Coullet-Tresser theory far away from the fixed map, we define:

Definition 2.13 If the map $\psi \in C^r(\mathcal{D})$ is infinitely doubling renormalizable and some of its renormalized maps belongs to the local stable manifold $W^s(\phi_0)$ of the fixed map ϕ_0, then its invariant Cantor set K, described in the theorem 2.12, is called a *Feigenbaum attractor.*

3 Persistence in one-parameter families

In this section we define a topology in the space of differentiable one-parameter families of maps in $C^r(\mathcal{D})$ and define persistence of the Feigenbaum attractor in one-parameter families near a given family.

Let $r \geq 1$. Let I be a closed interval such that $0 \in \text{int}I$.

Definition 3.1 $\mathcal{F}_I$ is the set of all one-parameter families $\Psi = \{\psi_t\}_{t \in I}$ such that

- for each fixed $t \in I$ the map ψ_t belongs to the space $C^r(\mathcal{D})$,
- the transformation that associates to each $t \in I$ the map $\psi_t \in C^r(\mathcal{D})$ is of class C^1.

The derivative in $C^r(\mathcal{D})$ respect to the parameter t is a vector in the Banach space of C^r maps from $\mathcal{D}$ to R^n. It is denoted as $\frac{\partial}{\partial t}\psi_t$.

The *topology in* $\mathcal{F}_I$ is given by the norm

$$\|\Psi\|_{1,r} = \max_{t \in I}\left\{\|\psi_t\|_r, \left\|\frac{\partial}{\partial t}\psi_t\right\|_r\right\}$$

Given $\varepsilon > 0$, we say that the family $X \in \mathcal{F}_I$ is ε-*close to* Ψ, if $\|X - \Psi\|_{1,r} < \varepsilon$, and we denote it as $X \in B_\varepsilon(\Psi)$.

Let us fix a family $\Psi \in \mathcal{F}_I$ such that ψ_0 has a Feigenbaum attractor K and $\frac{\partial}{\partial t}\psi_t\big|_{t=0} \neq 0$

We have the following classical theorem that uses the inverse function theorem in Banach spaces [La 1972].

Theorem 3.2 (Local immersion form) *If* $\frac{\partial}{\partial t}\psi_t|_{t=0} \neq 0$ *there exist a real number* $\delta > 0$, *a neighborhood* $\mathcal{U}$ *of* ψ_0 *in* $C^r(\mathcal{D})$, *a codimension one subspace* S *of the Banach space of* C^r *maps from* $\mathcal{D}$ *to* R^n, *and a* C^1 *diffeomorphism* $\wedge : \mathcal{U} \mapsto \hat{\mathcal{U}} \subset R \times S$ *such that* $\hat{\psi}_t = (t, 0)$ $\forall t \in (-\delta, \delta)$.

Remark 3.3 (Notation) Let us take a closed interval J contained in $(-\delta, \delta)$, with δ as in the theorem above, and such that $0 \in \text{int } J$. Note that if $|t_0|$ is small enough then $\hat{\psi}_{t+t_0} = \hat{\psi}_t + \hat{\psi}_{t_0}$ for all $t \in J$.

Through this section the Banach space $\mathcal{F}_J$ will be denoted simply as $\mathcal{F}$ and the restricted family $\{\psi_t\}_{t \in J}$ will be denoted as Ψ. To avoid confusion we will not use in this section the whole given family $\{\psi_t\}_{t \in I}$.

It is easy to verify that given $\varepsilon > 0$, there exists $\rho(\varepsilon) > 0$ such that, if $|t_0| < \rho(\varepsilon)$, then the reparametrized family

$$(+t_0)^*\Psi = \{\psi_{t+t_0}\}_{t \in J}$$

is ε- close to Ψ in $\mathcal{F}$.

Definition 3.4 *The Feigenbaum attractor is persistent in one-parameter families near* Ψ if there exist $\varepsilon > 0$ and a C^1 real function $a : B_\varepsilon(\Psi) \mapsto J$, such that, if $X = \{\chi_t\}_{t \in J} \in B_\varepsilon(\Psi) \subset \mathcal{F}$, then $\chi_{a(X)}$ exhibits a Feigenbaum attractor, $a(\Psi) = 0$ and $a((+t_0)^*\Psi) = -t_0$.

Theorem 3.5 *The Feigenbaum attractor is persistent in one parameter families near* Ψ *if and only if there exists in* $C^r(\mathcal{D})$ *a* C^1 *codimension-one local manifold* $\mathcal{M}$ *intersecting transversally the family* Ψ *at the map* ψ_0 *and such that any map* χ *in* $\mathcal{M}$ *has a Feigenbaum attractor.*

Proof:

It is easy to prove that the existence of $\mathcal{M}$ is sufficient for the persistence of the Feigenbaum attractor. In fact, the transversal intersection of Ψ and $\mathcal{M}$ at ψ_0 in the space $C^r(D)$, persists for any other family X sufficiently near Ψ in the space $\mathcal{F}$.

Let us prove the converse:

Let ε be the positive real number existing by the definition 3.4 of persistence. Let $0 < \rho < \rho(\varepsilon)$ as in the remark 3.3.

Let $\wedge : \mathcal{U} \mapsto \hat{\mathcal{U}}$ be the C^1 diffeomorphism of the local immersion form in theorem 3.2. As $\{\hat{\psi}_t : t \in J\}$ is a compact segment in $\hat{\mathcal{U}}$, there exists γ such that $\|\chi_0 - \psi_0\|_r < \gamma$ implies $\hat{\chi}_0 - \hat{\psi}_0 + \hat{\psi}_t \in \hat{\mathcal{U}}$ for all $t \in J$. Let us denote $B_\gamma(\psi_0) = \{\chi_0 \in C^r(\mathcal{D}) : \|\chi_0 - \psi_0\|_r < \gamma\}$.

Given $\chi_0 \in B_\gamma(\psi_0)$, define the family $X = \{\chi_t\}_{t \in J} \in \mathcal{F}$ translating $\hat{\psi}_t$ by the vector $\hat{\chi}_0$ in $\hat{\mathcal{U}}$. Precisely, $\chi_t \in \mathcal{U}$ is the preimage by the diffeomorphism $\wedge : \mathcal{U} \mapsto \hat{\mathcal{U}}$ of $\hat{\chi}_t$ defined as

$$\hat{\chi}_t = \hat{\psi}_t + \hat{\chi}_0$$

for all $t \in J$.

As it is a translation up to the C^1 diffeomorphism $\wedge$, the application $\mathcal{A}$ transforming $\chi_0 \in B_\gamma(\psi_0) \subset \mathcal{U} \subset C^r(\mathcal{D})$ into the family $X = \{\chi_t\}_{t \in J} \in \mathcal{F}$ is of C^1 class.

Besides $\mathcal{A}(\psi_0) = \Psi$ and $\mathcal{A}(\psi_{t_0}) = (+t_0)^*\Psi$, if $|t_0| < \rho(\varepsilon)$.

Thus, for small γ, the family X is ε-near Ψ. Using the definition 3.4 take the real function a and construct the C^1 real function $b : B_\gamma(\chi_0) \mapsto R$ defined as

$$b(\chi_0) = a(\mathcal{A}(\chi_0))$$

By the definition 3.4 $\chi_{b(\chi_0)}$ has a Feigenbaum attractor. Besides $b(\psi_0) = a(\Psi) = 0$ and

$$b(\psi_{t_0}) = a((+t_0)^*\Psi) = -t_0$$

if $t_0 < \rho(\varepsilon)$.

Differentiating the last identity respect to t_0 at $t_0 = 0$ we obtain

$$Db|_{\psi_0} \cdot \left.\frac{\partial \psi_t}{\partial t}\right|_{t=0} = -1$$

Thus $Db|_{\psi_0} \neq 0$ and, for small γ, the equality

$$\mathcal{M} = \{\chi_0 \in B_\gamma(\psi_0) : b(\chi_0) = 0\}$$

defines a local immersed submanifold in $C^r(\mathcal{D})$ passing through ψ_0, and transversal to Ψ.

As $b(\chi_0) = 0$ implies $\chi_{b(\chi_0)} = \chi_0$, and $\chi_{b(\chi_0)}$ has a Feigenbaum attractor, all χ_0 in $\mathcal{M}$ have such attractors.

■

The theorem 3.5 allows us to say that the existence of a Feigenbaum attractor is locally a codimension-one phenomenon in the space of C^r maps near ψ_0 when it is persistent in one-parameter families near Ψ.

Observe that Theorem 3.5 does not assert that *all* the maps near ψ_0 that exhibit a Feigenbaum attractor form a codimension one local manifold. We just constructed a local manifold $\mathcal{M}$ whose points are *some* of the maps having such attractors.

4 Proof of Theorem 1

To prove Theorem 1 we are fixing $r \geq 8$. Let $\mathcal{F} = \mathcal{F}_I$ be the Banach space of one-parameter families defined in 3.1.

We need to establish some new notation and to prove a lemma:

Remark 4.1 (and Notation) Let $\Psi = \{\psi_a\}_{a \in I}$ be a family in $\mathcal{F}$ such that ψ_0 is N times renormalizable. Thus $\mathcal{R}^N \psi_0 = \xi_N^{-1} \circ \psi_0^{2^N} \circ \xi_N$ for some C^r diffeomorphism ξ_N.

As the assumption of being N times renormalizable is an open condition in the space $C^r(\mathcal{D})$, there exist $\epsilon > 0$ and a neighborhood $\mathcal{V}$ of Ψ in $\mathcal{F}$, such that for all $X = \{\chi_a\}_{a \in I} \in \mathcal{V}$, χ_a is N times renormalizable if $a \in [-\epsilon, \epsilon]$, and $\mathcal{R}^N \chi_a = \xi_N^{-1} \circ \chi_a^{2^N} \circ \xi_N$ with ξ_N fixed.

Denote

$$\tilde{X} = \{\mathcal{R}^N \chi_a\}_{-\epsilon \leq a \leq \epsilon}$$

It is a one-parameter family of maps in $C^r(\mathcal{D})$, but it is not necessarily a C^1 curve of maps, because the renormalization is not differentiable in $C^r(\mathcal{D})$.

This pathology of the renormalization comes from the fact that the composition transforming $\chi \in C^r(\mathcal{D})$ into $\chi \circ \chi \in C^r(\mathcal{D})$, is continuous but not differentiable. In fact, the linear part of the increment $(\chi + h) \circ (\chi + h) - \chi \circ \chi$ can be shown to be $h \circ \chi + (D\chi \circ \chi) \cdot h$, working in $C^j(\mathcal{D})$ for $j \leq r - 1$. This says that the composition is differentiable from $C^r(\mathcal{D})$ to $C^j(\mathcal{D})$ if $j \leq r - 1$, but not in $C^r(\mathcal{D})$, because the difference $h \circ (\chi + h) - h \circ \chi$ does not decrease in $C^r(\mathcal{D})$ faster than h does.

Besides we would like more than $\mathcal{R}^N$ being differentiable in the space of maps. We would need that the application from a family X in $\mathcal{F}$ to the renormalized family $\tilde{X}$ be differentiable.

To obtain these good properties of the renormalization we shall work with the C^{r-2} topology in the set of the renormalized families.

Definition 4.2 The space $\tilde{\mathcal{F}}_\epsilon$ is the set of one-parameter families $\tilde{X} = \{\tilde{\chi}_a\}_{-\epsilon \leq a \leq \epsilon}$ such that

- for each $a \in [-\epsilon, \epsilon]$ the map $\tilde{\chi}_a$ is in $C^{r-2}(\mathcal{D})$.
- the transformation associating to each a the map $\tilde{\chi}_a \in C^{r-2}(\mathcal{D})$ is of class C^1.

In $\tilde{\mathcal{F}}_\epsilon$ consider the *topology* given by the norm

$$\|\tilde{X}\|_{1,r-2} = \max_{-\epsilon \leq a \leq \epsilon} \left\{ \|\tilde{\chi}_a\|_{r-2}, \left\| \frac{\partial}{\partial a} \tilde{\chi}_a \right\|_{r-2} \right\}$$

The following lemma gives the reason to work with $r - 2$ (instead of r) after renormalizing. And it is why we work with $r \geq 8$ instead of $r \geq 6$, increasing the differentiability by two orders from the value obtained in the theorem 2.10.

Lemma 4.3 *Let* $\Psi, \mathcal{V}, \epsilon, X$ *and* $\tilde{X}$ *as in the* Remark 4.1. *Let* $\mathcal{T}$ *be defined as* $\mathcal{T}(X) = \tilde{X}$ *for* $X \in \mathcal{V}$. *Then* $\mathcal{T} : \mathcal{V} \subset \mathcal{F} \mapsto \tilde{\mathcal{F}}_\epsilon$ *is of* C^1 *class.*

Proof:
$\tilde{X} = \{\mathcal{R}^N \chi_a\}_{-\epsilon \leq a \leq \epsilon}$ is a one-parameter family of maps in $C^r(\mathcal{D}) \subset C^{r-2}(\mathcal{D})$ and

$$\frac{\partial}{\partial a} \mathcal{R}^N \chi_a = \frac{\partial}{\partial a} \xi_N^{-1} \circ \chi_a^{2^N} \circ \xi_N = D\xi_N^{-1} \cdot \frac{\partial}{\partial a} \chi_a^{2^N} \circ \xi_N =$$

$$= D\xi_N^{-1} \cdot \left(\sum_{j=0}^{2^N-1} D\chi_a^j \cdot \left(\frac{\partial \chi_a}{\partial a} \circ \chi_a^{2^N-j-1} \circ \xi_N \right) \right) \in C^{r-1}(\mathcal{D}) \subset C^{r-2}(\mathcal{D})$$

In the equality above $D\chi_a^0$ and χ_a^0 are the identity.

Thus $\mathcal{T}(X)$ is a family in $\tilde{\mathcal{F}}_\epsilon$.

To see that $\mathcal{T}$ is of C^1 class let us compute the increment $\mathcal{T}(X+U) - \mathcal{T}(X)$ where $U = \{u_a\}_{a \in I}$ is near 0 in $\mathcal{F}$ so that $X + U \in \mathcal{V}$.

$$\mathcal{T}(X+U) - \mathcal{T}(X) = \{\xi_N^{-1} \circ (\chi_a + u_a)^{2^N} \circ \xi_N - \xi_N^{-1} \circ \chi_a^{2^N} \circ \xi_N\}_{-\epsilon \leq a \leq \epsilon}$$

For each fixed a, the map

$$\xi_N^{-1} \circ (\chi_a + u_a)^{2^N} \circ \xi_N - \xi_N^{-1} \circ \chi_a^{2^N} \circ \xi_N$$

can be written as

$$\sum_{j=0}^{2^N-1} D\xi_N^{-1}(\chi_a^{2^N} \circ \xi_N) \cdot D\chi_a^j(\chi_a^{2^N-j} \circ \xi_N) \cdot u_a(\chi_a^{2^N-j-1} \circ \xi_N) + \theta_a$$

where θ_a is the appropriate difference, between the increment and a linear part. This linear part is the candidate to be the differential of $\mathcal{T}$. It depends continuously on the given family $X \in \mathcal{F}$, because composition and multiplication are continuous in the space of families of finite differentiable maps.

Observe that $\theta_a \in C^{r-1}(\mathcal{D}) \subset C^{r-2}(\mathcal{D})$.

To prove that $\mathcal{T}$ is of C^1 class it is enough to prove that the family $\Theta = \{\theta_a\}_{-\epsilon \leq a \leq \epsilon}$ is in $\tilde{\mathcal{F}}_\epsilon$ and that

$$\frac{\|\Theta\|_{1,r-2}}{\|U\|_{1,r}} \to 0 \text{ when } \|U\|_{1,r} \to 0$$

We have that $\theta_a = \sum_{j=0}^{2^N-1}(A_j - L_j)$, where

$$\begin{aligned}
A_j &= \xi_N^{-1} \circ \chi_a^j \circ (\chi_a + u_a)^{2^N-j} \circ \xi_N - \xi_N^{-1} \circ \chi_a^{j+1} \circ (\chi_a + u_a)^{2^N-j-1} \circ \xi_N \\
L_j &= M_j \cdot u_a(\chi_a^{2^N-j-1} \circ \xi_N) \\
M_j &= D\xi_N^{-1}(\chi_a^{2^N} \circ \xi_N) \cdot D\chi_a^j(\chi_a^{2^N-j} \circ \xi_N)
\end{aligned}$$

Denoting

$$F_j(t) = \xi_N^{-1} \circ \chi_a^j \circ (\chi_a + tu_a) \circ (\chi_a + u_a)^{2^N-j-1} \circ \xi_N$$

for $t \in [0,1]$, we can write

$$A_j = F_j(1) - F_j(0) = \int_0^1 \frac{\partial}{\partial t} F_j(t)\ \mathrm{d}t \tag{1}$$

The function inside the integral is continuous from $[0,1]$ to $C^{r-1}(\mathcal{D})$. In fact, the derivative respect to t in $C^{r-1}(\mathcal{D})$, with fixed a, is:

$$\frac{\partial}{\partial t} F_j(t) = D\xi_N^{-1}(\chi_a^j \circ (\chi_a + tu_a) \circ (\chi_a + u_a)^{2^N-j-1} \circ \xi_N) \cdot v_j$$

where

$$v_j = D\chi_a^j((\chi_a + tu_a) \circ (\chi_a + u_a)^{2^N-j-1} \circ \xi_N) \cdot u_a((\chi_a + u_a)^{2^N-j-1} \circ \xi_N)$$

The equality (1) is also valid in $C^{r-2}(\mathcal{D})$. Therefore

$$\|A_j - L_j\|_{r-2} \le \int_0^1 \left\| \frac{\partial}{\partial t} F_j(t) - L_j \right\|_{r-2} \mathrm{d}t$$

Comparing $\frac{\partial}{\partial t}F_j(t)$ with L_j we can decompose its difference in three terms, as follows:

$$\begin{aligned} &\frac{\partial}{\partial t} F_j(t) - L_j = \\ &= \Delta(D\xi_N^{-1})v_j + D\xi_N^{-1}(\chi_a^{2^N} \circ \xi_N) \cdot \Delta(D\chi_a^j) \cdot u_a((\chi_a + u_a)^{2^N-j-1} \circ \xi_N) + M_j \Delta u_a \end{aligned} \tag{2}$$

where

$$\begin{aligned} \Delta(D\xi_N^{-1}) &= D\xi_N^{-1}(\chi_a^j \circ (\chi_a + tu_a) \circ (\chi_a + u_a)^{2^N-j-1} \circ \xi_N) - D\xi_N^{-1}(\chi_a^{2^N} \circ \xi_N) \\ \Delta(D\chi_a^j) &= D\chi_a^j((\chi_a + tu_a) \circ (\chi_a + u_a)^{2^N-j-1} \circ \xi_N) - D\chi_a^j(\chi_a^{2^N-j} \circ \xi_N) \\ \Delta u_a &= u_a((\chi_a + u_a)^{2^N-j-1} \circ \xi_N) - u_a(\chi_a^{2^N-j-1} \circ \xi_N) \end{aligned}$$

As the composition is continuous in $C^{r-2}(\mathcal{D})$, given $\delta > 0$ we obtain

$$\begin{aligned} \|\Delta(D\xi_N^{-1})\|_{r-2} &\le \delta \\ \|\Delta(D\chi_a^j)\|_{r-2} &\le \delta \end{aligned}$$

if $\|u_a\|_{r-2}$ is small enough.

Now, to obtain

$$\|\Delta u_a\|_{r-2} \le \delta \|u_a\|_{r-2}$$

it is sufficient that $\|u_a\|_{r-1}$ be small enough, because the $r-2$ first derivatives of u_a are Lipschitz with constant $\|u_a\|_{r-1}$.

Therefore, given $\delta > 0$

$$\|A_j - L_j\|_{r-2} \le \delta \|u_a\|_{r-2}, \quad \text{if } \|u_a\|_{r-1} \text{ is small enough.}$$

and so, given $\delta > 0$

$$\|\theta_a\|_{r-2} \le \delta \|u_a\|_{r-2}, \quad \text{if } \|u_a\|_{r-1} \text{ is small enough.} \tag{3}$$

Up to the moment we have proved that the renormalization $\mathcal{R}^N$ is differentiable from $C^{r-1}(\mathcal{D})$ to $C^{r-2}(\mathcal{D})$. We could have done the same from $C^r(\mathcal{D})$ to $C^{r-1}(\mathcal{D})$. We have loosen only one order in the differentiability of the maps. But we are going to loose another order when moving the parameter a of the family of C^r maps, and computing how the derivative respect to a behaves when renormalizing the family.

Let us compute the derivatives respect to a:

$$\frac{\partial \theta_a}{\partial a} = \sum_{j=0}^{2^N-1} \frac{\partial}{\partial a}(A_j - L_j)$$

We can decompose

$$\frac{\partial L_j}{\partial a} = B_j + C_j + E_j$$

where

$$
\begin{aligned}
B_j &= D^2\xi_N^{-1}(\chi_a^{2^N}\circ\xi_N)\cdot\left(D\chi_a^j(\chi_a^{2^N-j}\circ\xi_N)\cdot u_a(\chi_a^{2^N-j-1}\circ\xi_N),\ \frac{\partial}{\partial a}\chi_a^{2^N}\circ\xi_N\right)\\
C_j &= D\xi_N^{-1}(\chi_a^{2^N}\circ\xi_N)\cdot\frac{\partial}{\partial a}\left(D\chi_a^j(\chi_a^{2^N-j}\circ\xi_N)\right)\cdot u_a(\chi_a^{2^N-j-1}\circ\xi_N)\\
E_j &= D\xi_N^{-1}(\chi_a^{2^N}\circ\xi_N)\cdot D\chi_a^j(\chi_a^{2^N-j}\circ\xi_N)\cdot\frac{\partial}{\partial a}\left(u_a(\chi_a^{2^N-j-1}\circ\xi_N)\right)
\end{aligned}
$$

Observe that the maps above belong to $C^{r-2}(\mathcal{D})$.

Now let us compute the derivative of A_j:

$$
\frac{\partial A_j}{\partial a}=\frac{\partial}{\partial a}(\xi_N^{-1}\circ\chi_a^j\circ(\chi_a+u_a)^{2^N-j}\circ\xi_N-\xi_N^{-1}\circ\chi_a^{j+1}\circ(\chi_a+u_a)^{2^N-j-1}\circ\xi_N)
$$

Calling

$$
G_j(t)=D\xi_N^{-1}(\chi_a^j\circ(\chi_a+tu_a)\circ(\chi_a+u_a)^{2^N-j-1}\circ\xi_N)\cdot\frac{\partial}{\partial a}(\chi_a^j\circ(\chi_a+tu_a)\circ(\chi_a+u_a)^{2^N-j-1}\circ\xi_N)
$$

we can write

$$
\frac{\partial A_j}{\partial a}=G_j(1)-G_j(0)=\int_0^1\frac{\partial G_j(t)}{\partial t}\ \mathrm{d}t
$$

where the integral is computed in the Banach space $C^{r-2}(\mathcal{D})$, for each fixed a.

We can decompose $\frac{\partial}{\partial t}G_j(t)$ in three terms to be compared with B_j, C_j and E_j. In fact, denoting

$$
r_a=(\chi_a+u_a)^{2^N-j-1}\circ\xi_N
$$

we obtain

$$
\frac{\partial G_j(t)}{\partial t}=\hat{B}_j+\hat{C}_j+\hat{E}_j
$$

where

$$
\begin{aligned}
\hat{B}_j &= D^2\xi_N^{-1}(\xi_N\circ F_j(t))\cdot\left(D\chi_a^j((\chi_a+tu_a)\circ r_a)\cdot u_a(r_a),\ \frac{\partial}{\partial a}(\chi_a^j\circ(\chi_a+tu_a)\circ r_a)\right)\\
\hat{C}_j &= D\xi_N^{-1}(\xi_N\circ F_j(t))\cdot\frac{\partial}{\partial a}\left(D\chi_a^j((\chi_a+tu_a)\circ r_a)\right)\cdot u_a(r_a)\\
\hat{E}_j &= D\xi_N^{-1}(\xi_N\circ F_j(t))\cdot D\chi_a^j((\chi_a+tu_a)\circ r_a)\cdot\frac{\partial}{\partial a}\left(u_a(r_a)\right)
\end{aligned}
$$

To compute $\hat{C}_j$ and $\hat{E}_j$ we have differentiated first respect to t and then respect to a.

Joining all together we have that

$$
\frac{\partial}{\partial a}(A_j-L_j)=\int_0^1(\hat{B}_j-B_j+\hat{C}_j-C_j+\hat{E}_j-E_j)\ \mathrm{d}t
$$

Comparing $\hat{B}_j$ with B_j, both have u_a as a factor, evaluated at different points. With the same arguments used to analyze the equality (2), we conclude that, given $\delta>0$

$$
\|\hat{B}_j-B_j\|_{r-2}\le\delta\|u_a\|_{r-2},\quad\text{if }\|u_a\|_{r-1}\text{ is small enough.}
$$

Analogously

$$
\|\hat{C}_j-C_j\|_{r-2}\le\delta\|u_a\|_{r-2},\quad\text{if }\|u_a\|_{r-1}\text{ is small enough.}
$$

Comparing $\hat{E}_j$ with E_j, both have $\frac{\partial u_a}{\partial a}$ as a factor, also evaluated at different points. Therefore

$$\|\hat{E}_j - E_j\|_{r-2} \leq \delta \max\left\{\|u_a\|_{r-2}, \left\|\frac{\partial}{\partial a}u_a\right\|_{r-2}\right\},$$

if $\|u_a\|_{r-1}$ and $\left\|\frac{\partial}{\partial a}u_a\right\|_{r-1}$ are small enough.

We conclude that, given $\delta > 0$

$$\left\|\frac{\partial \theta_a}{\partial a}\right\|_{r-2} \leq \delta \max\left\{\|u_a\|_{r-2}, \left\|\frac{\partial}{\partial a}u_a\right\|_{r-2}\right\}, \tag{4}$$

if $\|u_a\|_{r-1}$ and $\left\|\frac{\partial}{\partial a}u_a\right\|_{r-1}$ are small enough.

To end the proof, from the inequalities (3) and (4), we conclude that given $\delta > 0$,

$$\|\Theta\|_{1,r-2} \leq \delta\|U\|_{1,r-1} \leq \delta\|U\|_{1,r}$$

if $\|U\|_{1,r-1} \leq \|U\|_{1,r}$ are small enough, as wanted.

■

Proof of Theorem 1:

Let $r \geq 8$. Thus $r-2 \geq 6$ and the theorem 2.10 is valid in C^{r-2}.

If ψ_0 has a Feigenbaum attractor K, it is infinitely renormalizable and there exists a natural N such that $\mathcal{R}^N\psi_0$ belongs to the local stable manifold $W^s(\phi_0)$, of the fixed point ϕ_0.

As in the Remark 4.1, consider $\mathcal{V}$ and $\epsilon > 0$, and denote $\tilde{\Psi}$ to $\mathcal{T}(\Psi) \in \tilde{\mathcal{F}}_\epsilon$ where $\mathcal{T}$ is the transformation of the Lemma 4.3. By hypothesis $\tilde{\Psi}$ intersects transversally $W^s(\phi_0)$ at $\mathcal{R}^N\psi_0$. Therefore, by Theorem 3.5, the Feigenbaum attractor is persistent in one-parameter families of $\tilde{\mathcal{F}}_\epsilon$ near $\tilde{\Psi}$. There exists a neighborhood $\tilde{\mathcal{U}}$ of $\tilde{\Psi}$ in $\tilde{\mathcal{F}}_\epsilon$ such that any family $\tilde{X}$ in $\tilde{\mathcal{U}}$ intersects $W^s(\phi_0)$. Let $b(\tilde{X})$ be the parameter value in the family $\tilde{X}$ where this intersection occurs. The real function b is of C^1 class, defined in $\tilde{\mathcal{U}}$.

Denote $\mathcal{U} = \left\{X \in \mathcal{V} \text{ such that } \mathcal{T}(X) \in \tilde{\mathcal{U}}\right\}$. Define $a : \mathcal{U} \mapsto I$ by $a(X) = b(\mathcal{T}(X))$ for all $X \in \mathcal{U}$. The real function a is of C^1 class, because b and $\mathcal{T}$ so are.

For all $X = \{\chi_t\}_{t\in I}$ in the neighborhood $\mathcal{U}$ of Ψ there exists $a(X)$ such that $\mathcal{R}^N\chi_{a(X)} \in W^s(\phi_0)$. Thus $\chi_{a(X)}$ has a Feigenbaum attractor.

Therefore, by Definition 3.4, the Feigenbaum attractor K is persistent in one-parameter families near Ψ. The Proposition 3.5 implies the existence of the C^1 codimension-one manifold in $C^r(\mathcal{D})$ passing through ψ_0.

■

5 Construction of a transversal family

We proved in Theorem 1 the existence of a local codimension one manifold of transformations having Feigenbaum attractors. To do so we used a given one-parameter family Ψ verifying, by hypothesis, a transversal condition after renormalized. In order to prove Theorem 2 it is enough to construct a good family Ψ, passing through the given transformation ψ_0.

As the renormalization is not surjective, the existence of Ψ is not obvious. It will be constructed using the following

Lemma 5.1 *Let ϕ_0 be the fixed map defined in* 2.1*, for $n \geq 2$.*

There exists a real analytic map $w : \mathcal{D} \mapsto R^n$ such that $u : \mathcal{D} \mapsto R^n$, defined as $u(x) = D\phi_0(x) \cdot w(x)$ for all $x \in \mathcal{D}$, is transversal to $W^s(\phi_0)$ in ϕ_0.

Proof: Let f_0 be the unimodal real analytic map in the interval $[-1, 1]$, fixed by the doubling renormalization. It is symmetric, and can be written as $f_0(x) = g_0(x^2)$ for g_0 a real analytic diffeomorphism from $[0, 1]$ to $[-\lambda, 1]$.

The Feigenbaum-Coullet-Tresser theory in the interval states that f_0 is a hyperbolic fixed point of the doubling renormalization, when working in the space of real analytic maps. Its local unstable manifold is a differentiable curve $\{f_t\}_t$ of symmetric real analytic unimodal maps of the interval $[-1, 1]$:

$$f_t(x) = g_t(x^2)$$

for $\{g_t\}_t$ a differentiable family of real analytic diffeomorphisms from $[0, 1]$ to its image [EW 1987].

In [CEK 1981] the Feigenbaum-Coullet-Tresser theory is extended to n dimensions in the analytic topology. In that work, the fixed map by the doubling renormalization is

$$\varphi_0(x_1, x_2, \ldots, x_n) = (g_0(x_1^2 - \alpha x_n), 0, \ldots, 0)$$

for some fixed number $\alpha \neq 0$, as observed in Remark 2.11. The local unstable manifold is formed by the maps φ_t:

$$\varphi_t(x_1, x_2, \ldots, x_n) = (g_t(x_1^2 - \alpha x_n), 0, \ldots, 0)$$

We showed in [CE 1998] (with the same conventions as in the article [CEK 1981]) that the unstable manifold in the space of real analytic maps is still valid in the space $C^r(\mathcal{D})$, for r large enough.

Now, according to the definition 2.1, we call *fixed map in n dimensions* to:

$$\phi_0(x_1, x_2, \ldots, x_n) = (x_n, 0, \ldots, 0, g_0(x_n^2))$$

Obviously the n-dimensional disks where ϕ_0 and φ_0 are defined, are different, but according to Remark 2.11, there exists a diffeomorphism ξ between them, conjugating ϕ_0 and φ_0. In fact $\phi_0 = \xi \circ \varphi_0 \circ \xi^{-1}$ where

$$\xi(x_1, x_2, \ldots, x_n) = (x_1, \ldots, x_{n-1}, g_0(x_1^2 - \alpha x_n))$$

So, in our context, the local unstable manifold $W^u(\phi_0)$ of ϕ_0 by the doubling renormalization $\mathcal{R}$ in $C^r(\mathcal{D})$, is the differentiable family $\{\phi_t\}_t$ where $\phi_t = \xi \circ \varphi_t \circ \xi^{-1}$.

Computing explicitly:

$$\phi_t(x_1, x_2, \ldots, x_n) = \left(g_t(g_0^{-1}(x_n)), 0, \ldots, 0, g_0([g_t(g_0^{-1}(x_n))]^2)\right)$$

Call $u = \left.\frac{\partial \phi_t}{\partial t}\right|_{t=0}$. It is transversal to $W^s(\phi_0)$, because it is the tangent vector to $W^u(\phi_0)$ at ϕ_0.

Let us compute explicitly u:

$$u(x_1, x_2, \ldots, x_n) = \left(\left.\frac{\partial g_t}{\partial t}\right|_{t=0} (g_0^{-1}(x_n)), 0, \ldots, 0, g_0'(x_n^2) \cdot 2x_n \cdot \left.\frac{\partial g_t}{\partial t}\right|_{t=0} (g_0^{-1}(x_n)) \right)$$

Now let us compute $D\phi_0$:

$$D\phi_0(x_1,\ldots,x_n) = \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 0 \\ . & \ldots & . & . \\ . & \ldots & . & . \\ . & \ldots & . & . \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 2x_n g_0'(x_n^2) \end{bmatrix}$$

Taking

$$w(x_1,\ldots,x_n) = \left(0,0,\ldots,\left.\frac{\partial g_t}{\partial t}\right|_{t=0}(g_0^{-1}(x_n))\right)$$

it is obtained $u = D\phi_0 \cdot w$ as wanted.

■

Remark 5.2 The proof of former lemma does not work in dimension one. In fact, observe that the matrix $D\phi_0$ is never null. For maps in the interval, it should be substituted by the derivative of the fixed map $2xg_0'(x^2)$ that vanishes for $x = 0$. In dimension two or larger, the derivative of the fixed map gives a spatial direction along which good perturbations can be constructed. In dimension one this spatial direction does not exist.

Now let us show how the former lemma allows us to construct a good family Ψ of C^r maps passing through the given map ψ_0:

Lemma 5.3 *If $\psi_0 \in C^r(\mathcal{D})$ has a Feigenbaum attractor, then there exists a one-parameter family $\Psi = \{\psi_t\}_{t\in I} \in \mathcal{F}$ such that the renormalized family $\tilde{\Psi}$, defined as $\{\mathcal{R}^N\psi_t\}_{-\epsilon\leq t\leq\epsilon}$, for some ϵ small enough and some N large enough, intersects transversally $W^s(\phi_0)$ at $\mathcal{R}^N\psi_0$ in the space $C^{r-2}(\mathcal{D})$.*

Proof:

As ψ_0 has a Feigenbaum attractor $\mathcal{R}^N\psi_0 = \xi_N^{-1}\circ\psi_0^{2^N}\circ\xi_N$ is as near as wanted to ϕ_0 in $C^r(\mathcal{D})$, for all N large enough.

Let $w : \mathcal{D} \mapsto R^n$ of class C^r be as in the lemma 5.1. If N is large enough, the vector $u : \mathcal{D} \mapsto R^n$, defined as $u(x) = D(\mathcal{R}^N\psi_0)(x)\cdot w(x)$ is transversal to $W^s(\phi_0)$ in $\mathcal{R}^N\psi_0$ in the space $C^{r-2}(\mathcal{D})$.

By the chain rule:

$$D(\mathcal{R}^N\psi_0)(x)\cdot w(x) = D\xi_N^{-1}(\psi_0^{2^N}\circ\xi_N(x))\cdot D\psi_0^{2^N-1}(\psi_0\circ\xi_N(x))\cdot D\psi_0(\xi_N(x))\cdot D\xi_N(x)\cdot w(x)$$

Using the density of C^r maps in $C^{r-2}(\mathcal{D})$, let us choose $v : \mathcal{D} \mapsto R^n$ of class C^r such that, in the C^{r-2} topology is sufficiently near $(D\psi_0\circ\xi_N)\cdot D\xi_N\cdot w$, so that

$$D\xi_N^{-1}(\psi_0^{2^N}\circ\xi_N)\cdot D\psi_0^{2^N-1}(\psi_0\circ\xi_N)\cdot v$$

is still a transversal vector to $W^s(\phi_0)$ in $\mathcal{R}^N\psi_0$ in the space $C^{r-2}(\mathcal{D})$.

Construct a C^r map v_0 from $\mathcal{D}$ to R^n such that:

- $v_0(x) = v(\xi_N^{-1}(x))$ if $x \in \xi_N(\mathcal{D})$ and
- $v_0(x) = 0$ if x belongs to some small neighborhood U of $\cup_{j=1}^{2^N-1}\psi_0^j\circ\xi_N(\mathcal{D})$.

and take $\psi_t = \psi_0 + tv_0$ for $-\epsilon \leq t \leq \epsilon$ for $\epsilon > 0$ small enough so ψ_t is N times renormalizable.

Let us differentiate $\mathcal{R}^N\psi_t = \xi_N^{-1} \circ \psi_t^{2^N} \circ \xi_N$ respect to t at $t = 0$. Observe that, if $x \in U$ then $\psi_t(x) = \psi_0(x)$. So we have:

$$\frac{\partial}{\partial t}\mathcal{R}^N\psi_t\bigg|_{t=0} = D\xi_N^{-1}(\psi_0^{2^N} \circ \xi_N(x)) \cdot D\psi_0^{2^N-1}(\psi_0 \circ \xi_N(x)) \cdot v_0(\xi_N(x))$$

Being $v_0(\xi_N(x)) = v(x)$ for all $x \in \mathcal{D}$, we obtain that $\frac{\partial}{\partial t}\mathcal{R}^N\psi_t\big|_{t=0}$ is transversal, by construction, to $W^s(\phi_0)$ at $\mathcal{R}^N\psi_0$, proving the lemma.

■

Proof of Theorem 2:

It is a straightforward conclusion of Lemma 5.3 and Theorem 1.

■

Referencias

[Ca 1996] E. Catsigeras: *Cascades of period doubling bifurcations in n dimensions.* Nonlinearity 9 (1061–1070) 1996

[CE 1998] E. Catsigeras, H. Enrich: *Homoclinic tangencies near cascades of period doubling bifurcations.* Ann. de l'IHP. An. non lin., vol.15, No.3 (255–299) 1998

[CEK 1981] P. Collet, J. P. Eckmann, H. Koch: *Period doubling bifurcations for families of maps on R^n.* Journ. of Stat. Phys., 25, N.1 (1–14) 1981

[Co 1996] E. Colli: *Infinitely many coexisting strange attractors.* Tese de Doutorado do IMPA Rio de Janeiro. To appear in Ann. de l'IHP. 1996

[CT 1978] P. Coullet, C. Tresser: *Itérations d'endomorphismes et groupe de renormalisation.* C.R. Acad. Sci. Paris 287 (577–588) 1978

[Da 1996] A. M. Davie: *Period doubling for $C^{2+\epsilon}$ mappings.* Comm. Math. Phys. 176 No. 2 (261–272) 1996

[EW 1987] J. P. Eckmann, P. Wittwer: *A complete proof of the Feigenbaum conjectures.* Journ. of Stat. Physics 46, N.3/4 (455–475) 1987

[Fe 1978] M. J. Feigenbaum: *Quantitative universality for a class of nonlinear transformations.* Journ. Stat. Physics, vol 19 (25–52) 1978

[Fe 1979] M. J. Feigenbaum: *The universal metric properties of non-linear transformations.* Journ. Stat. Physics, vol 21 (669–706) 1979

[GST 1989] J. M. Gambaudo, S. van Strien, C. Tresser: *Hénon-like maps with strange attractors: there exist C^∞ Kupka-Smale diffeomorphisms on S^2 with neither sinks nor sources.* Nonlinearity 2 (287–304) 1989

[GT 1992] J. M. Gambaudo, C. Tresser: *Self similar constructions in smooth dynamics: rigidity, smoothness and dimension.* Comm. Math. Phys. 150 (45–58) 1992

[La 1972] S. Lang: *Differentiable Manifolds.* Addison- Wesley. 1972

[La 1982] O. Lanford III: *A computer assisted proof of the Feigenbaum's conjectures.* Bulletin of the A.M.S. 6 (427–434) 1982

[Ly 1999] M. Lyubich: *Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture.* Ann. of Math.To appear

[Mc 1994] C. McMullen: *Complex dynamics and renormalization.* Ann. of Math. Stud., vol. 135, Princeton Univ. Press, Princeton, NJ 1994

[MS 1993] W. de Melo, S. van Strien: *One-dimensional dynamics.* Springer-Verlag, Berlin 1993

[PT 1993] J. Palis, F. Takens: *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations.* Cambridge University Press 1993

[Su 1991] D. Sullivan: *Bounds, quadratic differentials and renormalization conjectures.* Mathematics into the Twenty First Century vol. 2 (Providence, RI, Am. Math. Soc.)1991

[YA 1983] J. A. Yorke, K. T. Alligood: *Cascades of period doubling bifurcations: a prerrequisite for horseshoes.* Bull. A.M.S. 9 (319–322) 1983