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Pesin Entropy Formula for C^1 Diffeomorphisms with Dominated Splitting

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Abstract

For any C^1 diffeomorphism with dominated splitting we consider a nonempty set of invariant measures which describes the asymptotic statistics of Lebesgue-almost all orbits. They are the limits of convergent subsequences of averages of the Dirac delta measures supported on those orbits. We prove that the metric entropy of each of these measures is bounded from below by the sum of the Lyapunov exponents on the dominating subbundle. As a consequence, if those exponents are non negative, and if the exponents on the dominated subbundle are non positive, those measures satisfy the Pesin Entropy Formula.

1 Introduction

As pointed out by [P84, BCS13] and other authors, there is a gap between the $C^{1+\theta}$ and the C^1 Pesin Theory. To find new results that hold for C^1 maps relatively recent research started assuming some uniformly dominated conditions (see [ABC11, BCS13, ST10, ST12, T02]).

Let us consider $f \in \mathrm{Diff}^1(M)$, where M is a compact and connected Riemannian manifold of finite dimension. We denote by $\mathcal{P}$ the set of all Borel probability measures endowed with the weak* topology, and by $\mathcal{P}_f \subset \mathcal{P}$ the set of f-invariant probabilities. We denote by m a normalized Lebesgue measure, i.e. $m \in \mathcal{P}$. For any $\mu \in \mathcal{P}_f$, the orbit of x is regular for μ-a.e. $x \in M$ (see for instance [BP07, Theorem 5.4.1]). We denote the Lyapunov exponents of the orbit of x by

$$\chi_1(x) \geq \chi_2(x) \geq \ldots \geq \chi_{\dim M}(x).$$

Let

$$\chi_i^+(x) := \max\{\chi_i(x), 0\}.$$

Theorem (Ruelle's Inequality) [R78]
For all $f \in \mathrm{Diff}^1(M)$ and for all $\mu \in \mathcal{P}_f$

$$h_\mu \leq \int \sum_{i=1}^{\dim M} \chi_i^+ \, d\mu,$$

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where h_μ denotes the metric theoretical entropy of μ.

Definition 1.1 *Let $f \in \mathrm{Diff}^1(M)$ and $\mu \in \mathcal{P}_f$. We say that μ satisfies the* Pesin Entropy Formula*, and write $\mu \in PF$, if*

$$h_\mu = \int \sum_{i=1}^{\dim M} \chi_i^+ \, d\mu.$$

We denote by m^u the Lebesgue measure along the unstable manifolds of the regular points for which positive Lyapunov exponents and unstable manifolds exist. We denote the (zero dimensional) unstable manifold of x by $\{x\}$, and in this case we have $m^u = \delta_x$. For any invariant measure μ for which local unstable manifolds exist μ-a.e. we denote by μ^u the conditional measures of μ along the unstable manifolds, after applying the local Rohlin decomposition [R62].

The following are well known results of the Pesin Theory under the hypothesis $f \in \mathrm{Diff}^{\,2}(M)$:

Pesin Theorem [P77, M81, BP07] *Let $\mu \in \mathcal{P}_f$ be hyperbolic (namely, $\chi_i(x) \neq 0$ for all i and for $\mu-$a.e. $x \in M$). If $\mu \ll m$ then $\mu^u \ll m^u$ and $\mu \in PF$.*

Ledrappier-Strelcyn-Young Theorem [LS82, LY85] *$\mu \in PF$ if and only if $\mu^u \ll m^u$.*

Still in the C^2-scenario, non uniformly partially hyperbolic diffeomorphisms possess invariant measures μ such that $\mu^u \ll m^u$; and hence $\mu \in PF$ (see for instance [BDV05, Theorem 11.16]).

The general purpose of this paper is to look for adequate reformulations of some of the above results which hold for all $f \in \mathrm{Diff}^1(M)$. That is, we would like to know when an invariant measure under $f \in \mathrm{Diff}^1(M)$ satisfies Pesin Entropy Formula.

We first recall some definitions and previous results taken from [CE11].

Definition 1.2 (Asymptotic statistics)

Fix $x \in M$. The *sequence of empirical probabilities* of x is $\{\sigma_{n,x}\}_{n\geq 1} \subset \mathcal{P}$, where

$$\sigma_{n,x} := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}.$$

The *$p\omega$-limit* of x is

$$p\omega(x) := \{\mu \in \mathcal{P} : \ \exists \ n_j \to +\infty \text{ such that } \lim_{j\to\infty} \sigma_{n_j,x} = \mu\} \subset \mathcal{P}_f$$

We say that $p\omega(x)$ describes *the asymptotic statistics* of the orbit of x.

Definition 1.3 (Basins of statistical attraction)

For any given $\mu \in \mathcal{P}$ the *basin of (strong) statistical attraction of μ* is

$$B(\mu) := \{x \in M : \ p\omega(x) = \{\mu\}\}.$$

Consider a metric in $\mathcal{P}$ that induces the weak* topology and denote it by dist*.

The *basin of ε-weak statistical attraction of μ* is

$$B_\varepsilon(\mu) := \{x \in M : \ \mathrm{dist}^*(p\omega(x), \mu) < \varepsilon\}.$$

Definition 1.4 (SRB, physical and SRB-like measures)

An invariant probability measure μ is called *SRB* (and we denote $\mu \in$SRB) if the local unstable manifolds exist μ- a.e. and $\mu^u \ll m^u$.

The probability measure μ is called *physical* if $m(B(\mu)) > 0$.

If $f \in C^{1+\theta}$ then any hyperbolic ergodic SRB measure is physical. Nevertheless, if $f \in C^1$, the definition of SRB measure may not be meaningful since there may not exist local unstable manifolds ([P84, BCS13]). However, it still makes sense to define when a measure is physical.

In the C^1-scenario, we call a probability measure μ *SRB-like or pseudo-physical* (and we denote $\mu \in$ SRB-like) if $m(B_\varepsilon(\mu)) > 0$ for all $\varepsilon > 0$.

It is standard to check that the set of SRB-like measures is independent of the metric dist* chosen in $\mathcal{P}$ and that it is contained in $\mathcal{P}_f$.

Remark 1.5 (Minimal description of the asymptotic statistics of the system)

Given $f\colon M \to M$, we say that a weak*-compact set $\mathcal{K} \subset \mathcal{P}$ *describes the asymptotic statistics* of Lebesgue-almost all orbits of f if $p\omega(x) \subset \mathcal{K}$ for Lebesgue-almost all $x \in M$.

Theorems 1.3 and 1.5 of [CE11] prove that, for any continuous map $f\colon M \mapsto M$ the set of SRB-like measures is nonempty, it contains $p\omega(x)$ for Lebesgue-almost all $x \in M$, and it is the minimal weak*-compact set $\mathcal{K} \subset \mathcal{P}$ such that $p\omega(x) \subset \mathcal{K}$ for Lebesgue-almost all $x \in M$. Therefore, the set of SRB-like measures *minimally describes the asymptotic statistics of Lebesgue-almost all orbits.*

Our focus is to find relations, for C^1 diffeomorphisms, between:

- Physical measures and, more generally, SRB-like measures.
- Invariant measures μ such that $\mu \in PF$.

Several interesting results were already obtained for $f \in \mathrm{Diff}^1(M)$. First, in [T02] Tahzibi proved the Pesin Entropy Formula for C^1-generic area preserving diffeomorphisms on surfaces. More recently, Qiu [Q11] proved that if f is a transitive Anosov, then C^1-generically there exists a unique μ satisfying Pesin Entropy Formula. Moreover μ is physical and mutually singular with respect to Lebesgue (cf. [AB06]). Finally, we cite:

Sun-Tian Theorem [ST12]: *If $f \in \mathrm{Diff}^1(M)$ has an invariant measure $\mu \ll m$, and if there exists a dominated splitting $E \oplus F$ μ-a.e. such that $\chi_{\dim(F)} \geq 0 \geq \chi_{\dim(F)+1}$, then $\mu \in PF$.*

To prove this theorem Sun and Tian use an approach introduced by Mañé [M81]. In that approach he gave a new proof of Pesin Entropy Formula for $f \in \mathrm{Diff}^{1+\theta}(M)$ and hyperbolic $\mu \ll m$. Mañé's proof does not directly require the absolute continuity of the invariant foliations. So, it is reasonable to expect that it is adaptable to the C^1-scenario.

We reformulate the technique of Mañé [M81] to obtain an exact lower bound of the entropy for non necessarily conservative $f \in \mathrm{Diff}^1(M)$, provided that there exists a dominated splitting.

Definition 1.6 (Dominated splitting)

Let $f\colon M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold. Let $TM = E \oplus F$ be a continuous and df-invariant splitting such that $\dim(E), \dim(F) \neq 0$. We call $TM = E \oplus F$ a *dominated splitting* if there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\|df^n|_{E_x}\| \left\|df^{-n}|_{F_{f^n(x)}}\right\| \leq C\lambda^n, \ \forall x \in M \text{ and } n \geq 1.$$

We will prove the following results:

Theorem 1 *Let $f \in Diff^{\,1}(M)$ with a dominated splitting $TM = E \oplus F$. Let μ be an SRB-like measure for f. Then:*

$$h_\mu(f) \geq \int \sum_{i=1}^{\dim F} \chi_i \, d\mu. \tag{1}$$

Corollary 2 *Under the hypothesis of Theorem 1, if $\chi_{\dim F} \geq 0 \geq \chi_{\dim F+1}$, then μ satisfies the Pesin Entropy Formula.*

The proof of Corollary 2 is immediate: inequality (1) and Ruelle's Inequality imply that μ satisfies Pesin Entropy Formula. Moreover, as said in Remark 1.5, the set of SRB-like measures is nonempty. So, under the hypothesis of Corollary 2, there are invariant measures that satisfy the Pesin Entropy Formula. Besides, they minimally describe the asymptotic statistics of Lebesgue-almost all orbits.

Note that according to Avila and Bochi result [AB06] the measures of Theorem 1 and Corollary 2 are C^1-generically mutually singular with respect to Lebesgue.

Remark 1.7 The same arguments of the proof of Theorem 1 also work under hypothesis that are more general than the global dominated splitting assumption. In fact, if $\Lambda \subset M$ is an invariant and compact topological attractor, and if $\overline{V} \supset \Lambda$ is a compact neighborhood with dominated splitting $T_{\overline{V}} = E \oplus F$, then the same statements and proofs of Theorem 1 and Corollary 2 hold for $f|_V$.

Now, let us pose an example for which Theorem 1 and Corollary 2 do not hold. Consider the simple eight-figure diffeomorphism in [BP07, Figure 10.1]. In this example, the Dirac-delta measure μ supported on a fixed hyperbolic point p is physical. Thus μ is SRB-like. Besides, there exists a dominated splitting μ-a.e. because p is hyperbolic. Nevertheless, inequality (1) does not hold because $h_\mu = 0$ and the Lyapunov exponent along the unstable subspace of $T_p(M)$ is strictly positive. So, the presence of a dominated splitting *just μ-a.e.* is not enough to obtain Theorem 1.

The following question arises from the statements of our results: Does the SRB-like property characterize all the measures that satisfy Pesin Entropy Formula? The answer is negative. In fact, the converse statement of Corollary 2 is false. As a counter-example consider a C^2 non transitive uniformly hyperbolic attractor, with a finite set $\mathcal{K} = \{\mu_1, \mu_2, \ldots, \mu_k\}$ ($k \geq 2$) of distinct SRB ergodic measures (hence each μ_i is physical) such that $\mathcal{K}$ statistically attracts Lebesgue-almost every orbit. Therefore, the set of all SRB-like measures coincides with $\mathcal{K}$ (see Remark 1.5). So, $(\mu_1 + \mu_2)/2 \notin \mathcal{K}$ is not an SRB-like measure. After Corollary 2, μ_1 and μ_2 satisfy

Pesin Entropy Formula. It is well known that any convex combination of measures that satisfy Pesin Entropy Formula also satisfies it (see Theorem 5.3.1 and Lemma 5.2.2. of [K98]). We conclude that $(\mu_1 + \mu_2)/2$ satisfies Pesin Formula but it is not SRB-like.

The paper is organized as follows: In Section 2 we reduce the proof of Theorem 1 to Lemmas 2.2 and 2.3. In Sections 3 and 4 we prove Lemmas 2.2 and 2.3 respectively. Finally, in Section 5 we check some technical assertions that are used in the proofs of the previous sections.

2 Reduction of the proof of Theorem 1

For the diffeomorphism $f\colon M \to M$ with dominated splitting $E \oplus F = TM$, we denote:

$$\psi(x) := -\log \left| \det df(x)|_{F_x} \right| \tag{2}$$

$$\psi_n(x) := -\log \left| \det df^n(x)|_{F_x} \right| = \sum_{j=0}^{n-1} \psi \circ f^j(x) = -\log \left| \det df^{-n}(f^n(x))|_{F_{f^n(x)}} \right| \tag{3}$$

Consider a metric dist^* in the space $\mathcal{P}$ of all Borel probability measures inducing its weak* topology. For all $\mu \in \mathcal{P}$, for all $\varepsilon > 0$ and for all $n \geq 1$, we denote:

$$C_n(\varepsilon) := \{x \in M : \ \mathrm{dist}^*(\sigma_{n,x}, \mu) < \varepsilon\}, \tag{4}$$

where $\sigma_{n,x}$ is the empirical probability according to Definition 1.2. We call $C_n(\varepsilon)$ the *approximation up to time n* of the basin $B_\varepsilon(\mu)$ of ε-weak statistical attraction of the measure μ (cf. Definition 1.3).

Proposition 2.1 *Let $f \in \mathrm{Diff}^1(M)$ with a dominated splitting $TM = E \oplus F$. There exists a weak* metric dist^* in $\mathcal{P}$, such that for any f-invariant probability measure μ the following inequality holds:*

$$\lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{\log m(C_n(\varepsilon))}{n} \leq h_\mu(f) + \int \psi \, d\mu, \tag{5}$$

where m is the Lebesgue measure.

We note that the term $h_\mu(f) + \int \psi \, d\mu$ is non negative due to Ruelle's Inequality. Nevertheless, it is bounded from below by inequality (5), which relates it with the Lebesgue measure m.

At the end of this section, we reduce the proof of Proposition 2.1 to Lemmas 2.2 and 2.3. Along the remaining sections we prove these two lemmas. Now, let us prove the following assertion:

Proposition 2.1 implies Theorem 1.

Proof:

Let μ be f-invariant. Assume that μ does not satisfy inequality (1). In other words,

$$h_\mu(f) + \int \psi \, d\mu = -r < 0.$$

From Proposition 2.1, for all $\varepsilon > 0$ small enough there exists $N \geq 1$ such that

$$\frac{\log m(C_n(\varepsilon))}{n} \leq \frac{-r}{2} \quad \forall\ n \geq N.$$

Since $r > 0$, we deduce that $\sum_{n=1}^{+\infty} m(C_n(\varepsilon)) < +\infty$. Thus, by Borel-Cantelli Lemma the set $\bigcap_{N\geq 1} \bigcup_{n\geq N} C_n(\varepsilon)$ has zero m-measure. By Definition 1.3 we have $B_\varepsilon(\mu) \subset \bigcap_{N\geq 1} \bigcup_{n\geq N} C_n(\varepsilon)$. So, $m(B_\varepsilon(\mu)) = 0$, and applying Definition 1.4 we conclude that μ is not SRB-like, proving Theorem 1. □

Proposition 2.1 follows from Lemmas 2.2 and 2.3.

To prove Proposition 2.1, we take from [ST12] the idea of using Mañé's approach [M81]. Nevertheless, we use this approach in a distinct context (i.e. we do not assume $\mu \ll m$) and apply different arguments. In [M81] Mañé considers $f \in C^{1+\theta}$ and constructs a C^1 foliation $\mathcal{L}$, which is not necessarily invariant, but approximates the unstable invariant foliation. On the one hand, the given invariant measure $\mu \ll m$ has absolutely continuous conditional measures along the leaves of $\mathcal{L}$, because m has. On the other hand, the hypothesis $f \in C^{1+\theta}$ allows Mañé to use the Bounded Distortion Lemma. So, he obtained Pesin Entropy Formula after taking $f^n\mathcal{L}$ convergent to the unstable foliation.

In our case these arguments fail to work, except one. There still exists a C^1 (non invariant) foliation $\mathcal{L}$ whose tangent sub-bundle approximates the dominating sub-bundle F. Besides, since $\mathcal{L}$ is C^1, the conditional measures of m (not of μ) along the leaves of $\mathcal{L}$ are absolutely continuous. But we have neither the hypothesis $\mu \ll m$ nor the $C^{1+\theta}$ regularity of f. Also an invariant foliation to which $f^n\mathcal{L}$ would converge, may fail to exist. The role of the following Lemmas 2.2 and 2.3 is to overcome these problems. Before stating them, we adopt the following:

Notation. Let $\mathcal{B}$ be the Borel σ-algebra on the manifold M. We denote by $\alpha = \{X_i\}_{1\leq i\leq k}$ a finite partition of M, namely:

$$\begin{gathered} X_i \in \mathcal{B} \text{ for all } \ 1 \leq i \leq k, \\ X_i \bigcap X_j = \emptyset \text{ if } i \neq j, \\ \textstyle\bigcup_{i=1}^{k} X_i = M. \end{gathered}$$

We write $f^{-j}(\alpha) = \{f^{-j}(X_i)\}_{1\leq i\leq k}$.

For any pair of finite partitions $\alpha = \{X_i\}_{1\leq i\leq k}$ and $\beta = \{Y_j\}_{1\leq j\leq h}$ we denote

$$\alpha \vee \beta = \{X_i \cap Y_j : 1 \leq i \leq k,\ 1 \leq j \leq h,\ X_i \cap Y_j \neq \emptyset\},$$

$$\alpha^n = \bigvee_{j=0}^{n} f^{-j}(\alpha).$$

Lemma 2.2 (Upper bound of the Lebesgue measure m)

For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite partition α with $\operatorname{diam}(\alpha) < \delta$ *there exist a sequence $\{\nu_n\}_{n\geq 0}$ of finite measures and a constant $K > 0$ such that:*

(i) *$\nu_n(X) < K$ for all $X \in \alpha^n = \bigvee_{j=0}^{n} f^{-j}(\alpha)$, for all $n \geq 0$.*

(ii) *The following inequality holds for all $C \in \mathcal{B}$ and for all $n \in \mathbb{N}^+$:*

$$m(C) \leq K e^{n\varepsilon} I(\psi_n, C, \nu_n), \quad \text{where}$$

$$I(\psi_n, C, \nu_n) := \int_C e^{\psi_n}\, d\nu_n. \tag{6}$$

We will prove Lemma 2.2 in Section 3.

Before stating the second lemma, recall equality (4).

Lemma 2.3 (Lower bound of the metric entropy)

There exists a metric dist^* *in $\mathcal{P}$ with the following property:*

For all $\mu \in \mathcal{P}_f$ and for all $\varepsilon,\ \delta > 0$ there exist a finite partition α satisfying $\operatorname{diam}\alpha < \delta$, *and a real number $\varepsilon_0^* > 0$ such that:*

For all $0 < \varepsilon^* < \varepsilon_0^*$, and for any sequence $\{\nu_n\}_{n\geq 0}$ of finite measures such that there exists $K > 0$ satisfying $\nu_n(X) < K$ for all $X \in \alpha^n$ for all $n \geq 0$, the following inequality holds:

$$\limsup_{n\to+\infty} \frac{1}{n} \log I(\psi_n, C_n(\varepsilon^*), \nu_n) \leq \varepsilon + h_\mu(\alpha) + \int \psi\, d\mu.$$

We will prove Lemma 2.3 in Section 4.

To end this section let us prove that Lemmas 2.2 and 2.3 imply Proposition 2.1:

Proof: Let $\mu \in \mathcal{P}_f$ and $\varepsilon > 0$. Consider $\delta > 0$ obtained from Lemma 2.2. Applying Lemma 2.3, construct the partition α, the number ε_0^* and the sequence $\{C_n(\varepsilon^*)\}_{n\geq 0} \subset \mathcal{B}$ for any $0 < \varepsilon^* < \varepsilon_0^*$.

Apply again Lemma 2.2 to obtain the sequence $\{\nu_n\}_{n\geq 0}$ of finite measures and the constant $K > 0$.

We now apply again Lemma 2.3 to deduce:

$$\limsup_{n\to\infty} \frac{1}{n} \log I(\psi_n, C_n(\varepsilon^*), \nu_n) \leq \varepsilon + h_\mu(\alpha) + \int \psi\, d\mu \ \ \forall\ 0 < \varepsilon^* < \varepsilon_0^* \tag{7}$$

Besides, by Lemma 2.2:

$$\frac{1}{n} \log m(C_n(\varepsilon^*)) \leq \frac{\log K}{n} + \varepsilon + \frac{1}{n} \log I(\psi_n, C_n(\epsilon^*), \nu_n). \tag{8}$$

We join the two inequalities (7) and (8) to deduce that:

$$\limsup_{n\to\infty} \frac{1}{n} \log m(C_n(\varepsilon^*)) \leq 2\,\varepsilon + h_\mu(\alpha) + \int \psi\, d\mu \ \ \forall\ 0 < \varepsilon^* < \varepsilon_0^*.$$

Taking $\varepsilon^* \to 0^+$ we obtain:

$$\lim_{\varepsilon^*\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log m(C_n(\varepsilon^*)) \leq 2\,\varepsilon + h_\mu(\alpha) + \int \psi\, d\mu.$$

Since $\varepsilon > 0$ is arbitrary, we deduce inequality (5), as wanted. □

3 Proof of Lemma 2.2

To prove Lemma 2.2 we will use the technique of the dispersion of Hadamard graphs, following Mañé in [M81].

NOTATION: First take a fixed value of $\delta > 0$ small enough such that $\exp_x^{-1}$ is a diffeomorphism from $B_{3\delta}(x)$ onto its image in T_xM for all $x \in M$. Fix $x \in M$. Denote

$$\mathbf{B}_\delta^{E_x}(\mathbf{0}) := \{v \in E_x : \|v\| \leq \delta\},$$

$$\mathbf{B}_\delta^{F_x}(\mathbf{0}) := \{v \in F_x : \|v\| \leq \delta\},$$

$$\mathbf{B}_\delta^{T_xM}(\mathbf{0}) := \mathbf{B}_\delta^{E_x}(\mathbf{0}) \oplus \mathbf{B}_\delta^{F_x}(\mathbf{0}).$$

Denote by π_{E_x} (resp. π_{F_x}) the projection of T_xM on E_x along F_x (resp. on F_x along E_x), and $\gamma := \max_{x\in M}\{\|\pi_{E_x}\|, \|\pi_{F_x}\|\}$.

For any $v \in \mathbf{B}_\delta^{T_xM}(\mathbf{0})$ we denote $v_1 := \pi_{E_x} v$, $v_2 := \pi_{F_x} v$.

Definition 3.1 G is a *Hadamard graph* (or simply "a graph") if

G: $\mathbf{B}_\delta^{E_x}(\mathbf{0}) \times \mathbf{B}_\delta^{F_x}(\mathbf{0}) \to \mathbf{B}_\delta^{E_x}(\mathbf{0})$,

$G(v_1, 0) = 0$ for all $v_1 \in \mathbf{B}_\delta^{E_x}(\mathbf{0})$ and

$\Phi(v_1, v_2) = v_1 + v_2 + G(v_1, v_2) \in \mathbf{B}_{2\delta}^{T_xM}(\mathbf{0})$ is a C^1-diffeomorphism onto its image. (See Figure 1.)

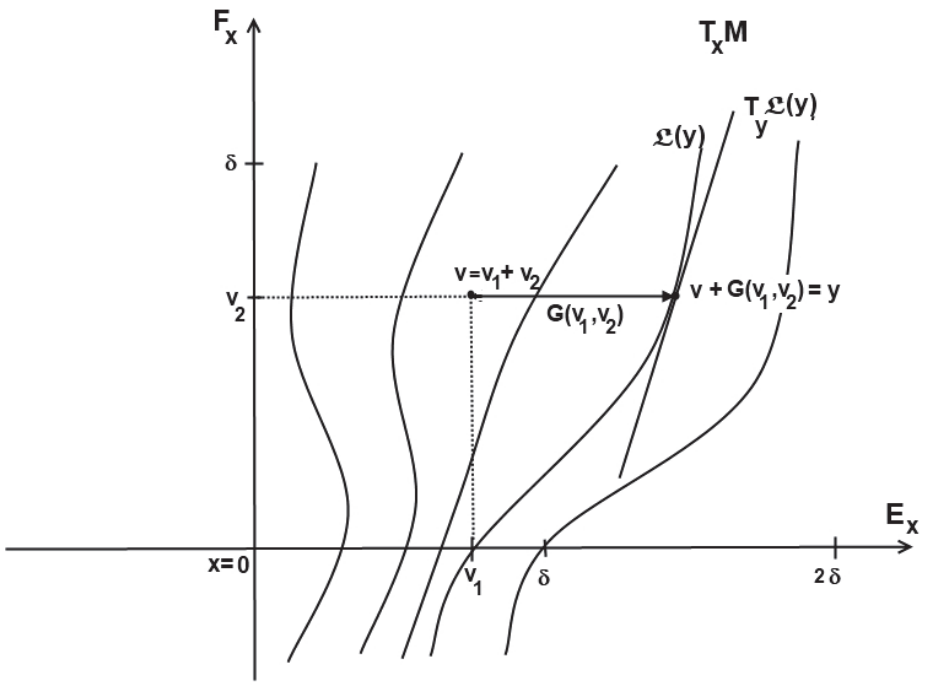

Figure 1: The foliation $\mathcal{L}$ associated to a Hadarmard graph (We omit the exponential map $\exp_x$)

The foliation $\mathcal{L}$ associated to the graph G is the foliation whose leaves are parametrized on $v_2 \in \mathbf{B}_\delta^{F_x}(\mathbf{0}) \subset F_x$, with constant v_1, by the diffeomorphism:

$$\exp_x(\Phi(v_1, \cdot)) = \exp_x(v_1 + \cdot + G(v_1, \cdot))$$

In Figure 1 we draw the foliation $\mathcal{L}$. To simplify the notation we omit the exponential map $\exp_x$ and denote $y = v_1 + v_2 + G(v_1, v_2)$. The leaf containing y is denoted by $\mathcal{L}(y)$.

Definition 3.2 DISPERSION OF **G**

The dispersion of the graph G is

$$\operatorname{disp} G = \max_{v \in \mathbf{B}_\delta^{T_x M}(\mathbf{0})} \left\{ \left\| \frac{\partial G}{\partial v_2}(v_1, v_2) \right\| \right\},$$

where $v = v_1 + v_2$ and $\partial G / \partial v_2$ denotes the Fréchet derivative of

$$G(v_1, \cdot) \colon \mathbf{B}_\delta^{F_x}(\mathbf{0}) \to \mathbf{B}_\delta^{E_x}(\mathbf{0})$$

with a constant value of $v_1 \in \mathbf{B}_\delta^{E_x}(\mathbf{0})$.

We denote by m^W the Lebesgue measure along an embedded local submanifold $W \subset M$.

Assertion 3.3

$$T_y\mathcal{L}(y) = \left(\operatorname{Id}|_{F_x} + \frac{\partial G}{\partial v_2} \right) F_x, \tag{9}$$

$$m^{\mathcal{L}(y)}(\mathcal{L}(y)) \leq [(1 + \operatorname{disp} G)\delta]^{\dim F}, \tag{10}$$

For all $\varepsilon > 0$ *there exists* $c > 0$ *such that, if* $\operatorname{disp} G \leq c$, *then*

$$\operatorname{dist}(T_y\mathcal{L}(y), F_x) < \frac{\varepsilon}{2} \qquad \forall\ y \in \operatorname{Im}(\Phi).$$

For such a value of $c > 0$ *(depending on* $\varepsilon > 0$*), there exists* $\delta_1 > 0$ *such that, if* $\operatorname{dist}(x, y) < \delta_1$, *then* $y \in \operatorname{Im}(\Phi)$ *and*

$$\operatorname{dist}(T_y\mathcal{L}(y), F_y) < \varepsilon \tag{11}$$

Proof: The assertion follows from the properties that were established in the definition of Hadamard graphs and their associated foliations, and from the definition of dispersion. In particular (11) holds because of the continuous dependence of the splitting $E_y \oplus F_y$ on the point y. □

3.4 Iterating the local foliation $\mathcal{L}$

Denote by $B^n_\delta(x)$ the dynamical ball defined as

$$B^n_\delta(x) := \{y \in M : \mathrm{dist}(f^j(x), f^j(y)) < \delta \ \forall\, 0 \le j \le n\}.$$

Take any graph G in $\mathbf{B}_\delta^{T_xM}(\mathbf{0})$ such that $\mathrm{disp}\, G < 1/2$, and consider its associated local foliation $\mathcal{L}$. Construct the image $f^n(\mathcal{L})$ in the dynamical ball $B^n_\delta(x)$, i.e.:

$$f^n(\mathcal{L} \cap B^n_\delta(x)) = f^n \exp_x(v_1 + v_2 + G(v_1, v_2))$$

for all $(v_1, v_2) \in \mathbf{B}_\delta^{E_x}(\mathbf{0}) \times \mathbf{B}_\delta^{F_x}(\mathbf{0})$ such that $\exp_x(v_1 + v_2 + G(v_1, v_2)) \in B^n_\delta(x)$.

Lemma 3.5 (Reformulation of Lemma 4 of [M81])
There exists $0 < c' < 1/2$ depending only on f, such that for all $0 < c < c'$ there are δ_0, $n_0 > 0$ such that for any point $x \in M$, if $\mathcal{L}$ is the local foliation associated to a graph G defined on T_xM with

$$\mathrm{disp}\ G < c,$$

then for all $n \ge 0$ the iterated foliation $f^n(\mathcal{L} \cap B^n_{\delta_0}(x))$ is contained in the associated foliation of a graph G_n defined on $T_{f^n(x)}M$, and

$$\mathrm{disp}\ G_n < c \text{ for all } n \ge n_0. \tag{12}$$

Besides, for all $y \in B^n_{\delta_0}(x)$ the image $f^n(\mathcal{L}(y) \cap B^n_{\delta_0}(x))$ is contained in a single leaf of the foliation associated to G_n.

Proof:
Step 1. Choose $n_0 \ge 0$ such that

$$\left\|df^n|_{E_z}\right\| \ \left\|df^{-n}|_{F_{f^n(z)}}\right\| < 1 \ \ \forall\ n \ge n_0, \ \ \forall\ x \in M. \tag{13}$$

For such a fixed value of n_0, take $\delta_0 > 0$ so that for all $x \in M$, for all $0 \le n \le n_0$, and for any graph G defined in $\mathbf{B}_\delta^{T_xM}(\mathbf{0})$ with $\mathrm{disp}\, G < 1/2$, there exists a graph G_n defined on $\mathbf{B}_\delta^{T_{f^n(x)}M}(\mathbf{0})$ satisfying the following condition:

$$\begin{array}{l} \text{for any } \ y = \exp_x(v_1 + v_2 + G(v_1, v_2)) \in B_{\delta_0}(x) \\ \text{there exists } (u_1, u_2) \in \mathbf{B}_\delta^{E_{f^n(x)}}(\mathbf{0}) \times \mathbf{B}_\delta^{F_{f^n(x)}}(\mathbf{0}) \\ \text{where } u_1 \text{ depends only on } v_1 \text{ and} \\ f^n(y) = \exp_{f^n(x)}(u_1 + u_2 + G_n(u_1, u_2)). \end{array} \tag{14}$$

In Assertion 5.1 of the appendix we show that such $\delta_0 > 0$ exists. We note that the above assertion is true for any initial graph G with dispersion smaller than $1/2$ and that Statement (14) a priori *only* holds if $0 \le n \le n_0$. The assertion that u_1 depends only on v_1 implies that the image $f^n(z)$ of any point $z \in B_{\delta_0}(x)$ in the leaf $\mathcal{L}(y)$ associated to the graph G, is contained in the leaf of $f^n(y)$ associated to the graph G_n.

STEP 2. With $\delta_0 > 0$ fixed as above, there exists $0 < c' < 1/2$ such that for any graph G with $\operatorname{disp} G < c'$ and for all $n \geq 0$, if G_n is the graph defined in $\mathbf{B}_\delta^{T_{f^n(x)}M}(\mathbf{0})$ satisfying (14), then:

$$\left\| \frac{\partial G_n}{\partial u_2}(u_1, u_2) \right\| \leq \|df^n|_{E_x}\| \cdot \operatorname{disp} G \cdot \left\|df^{-n}|_{F_{f^n(x)}}\right\| \quad \forall\ y \in B^n_{\delta_0}(x) \tag{15}$$

We prove this statement in Assertion 5.2 of the appendix.

STEP 3. Due to the construction of δ_0 in Step 1, inequality (15) holds in particular for $n = n_0$ for any G such that $\operatorname{disp} G < c'$. Therefore, using Inequalities (13) and (15) and Definition 3.2, we obtain:

$$\operatorname{disp} G_{n_0} \leq \operatorname{disp} G < c' \quad \forall\ G \text{ s.t. } \operatorname{disp} G < c',$$

Moreover, if $\operatorname{disp} G < c < c'$, then $\operatorname{disp} G_{n_0} < c < c'$.

STEP 4. From the construction of δ_0 in Step 1 and using that $\operatorname{disp} G_{n_0} < c < c' < 1/2$, we deduce that the graph G_n exists for all $n_0 \leq n \leq 2n_0$. Moreover, $\|df^n|_{E_x}\| \cdot \|df^{-n}|_{F_{f^n(x)}}\| < 1$ for all $n \geq n_0$. So, applying inequality (15) we obtain $\operatorname{disp} G_n < c$ for all $n_0 \leq n \leq 2n_0$. Finally, applying inductively Assertions (13) and (14) we conclude that the graph G_n exists for all $n \geq 0$ and $\operatorname{disp} G_n < c$ for all $n \geq n_0$. □

Once the constant c' of Lemma 3.5 is fixed, depending only on f, one obtains the following property that allows to move the reference point x (used to construct the graph G on $\mathbf{B}_\delta^{T_xM}(\mathbf{0})$), preserving the *same* associated local foliation $\mathcal{L}$ and the uniformity of the upper bound of its dispersion:

Lemma 3.6 *For all $0 < c < c'$ there exists $\delta_1 > 0$ such that, for any $x \in M$ and for any graph G with $\operatorname{disp}(G) < c/2$ defined in $\mathbf{B}_\delta^{T_xM}(\mathbf{0})$, the associated foliation $\mathcal{L}$ in the neighborhood $B_{\delta_1}(x)$ is also associated to a graph G' defined in $\mathbf{B}_\delta^{T_zM}(\mathbf{0})$ for any $z \in B_{\delta_1}(x)$. Besides $\operatorname{disp}(G') < c$.*

Proof: The splitting $E_z \oplus F_z$ depends continuously on $z \in M$. Then π_{E_z} and π_{F_z} also depend continuously on z. Therefore, for all $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$\|\pi_{F_x}|_{E_z}\| < \varepsilon, \qquad \|\pi_{E_x}|_{F_z}\| < \varepsilon \quad \text{ if } \operatorname{dist}(x, z) < \delta_1.$$

(For simplicity in the notation in the above inequalities we omit the derivative of $\exp_x^{-1} \circ \exp_z$ which identifies T_zM with T_xM.)

We claim that if $\delta_1 > 0$ is small enough then, for any graph G defined on $\mathbf{B}_\delta^{T_xM}(\mathbf{0})$, and for any point z such that $\operatorname{dist}(z, x) < \delta_1$, there exists a graph G' defined on $\mathbf{B}_\delta^{T_zM}(\mathbf{0})$ such that the local foliations associated to G and G' coincide in an open set where both are defined. In fact, G' should satisfy the following equations:

$$\begin{aligned} u_1 + u_2 + G'(u_1, u_2) &= v_1 + v_2 + G(v_1, v_2), \\ u_1, G'(u_1, u_2) \in E_z, \qquad u_2 \in F_z, \qquad G'(u_1, 0) &= 0. \end{aligned} \tag{16}$$

Since by hypothesis G is a graph, it is C^1 and

$$v_1, \quad G(v_1, v_2) \in E_x, \quad v_2 \in F_x, \quad G(v_1, 0) = 0.$$

The above equations are solved by

$$u_1 := \pi_{E_z}(v_1), \qquad u_2 := \pi_{F_z}(v_1 + v_2 + G(v_1, v_2)), \tag{17}$$

$$G' = -u_1 + \pi_{E_z}(v_1 + v_2 + G(v_1, v_2)). \tag{18}$$

The two equalities in (17) define a local diffeomorphism $\Psi(v_1, v_2) = (u_1, u_2)$. In fact, on the one hand $u_1 = \pi_{E_z}|_{E_x}(v_1)$, where $\pi_{E_z}|_{E_x}$ is a diffeomorphism (which is linear and uniformly near the identity map, independently of the graph G). On the other hand, for v_1 constant, the derivative with respect to v_2 of $\pi_{F_z}(v_1 + v_2 + G(v_1, v_2))$ is $\pi_{F_z}|_{T_y\mathcal{L}(y)}$, which is, independently of the graph G, uniformly near $\pi_{F_x}|_{T_y\mathcal{L}(y)} = Id|_{F_x}$. Thus, Ψ is a local diffeomorphism C^1 near the identity map provided that δ_1 is chosen small enough (independently of the given graph G).

From the above construction we deduce that the composition of the mapping $\Psi(u_1, u_2) = (v_1, v_2)$ with the mapping $(v_1, v_2) \mapsto G'$ defined by (18), is of C^1 class. Therefore $G'(u_1, u_2)$ is C^1 dependent on (u_1, u_2). Besides $G'(u_1, 0) = 0$ because $G(v_1, 0) = 0$. Due to Identity (16), the application ϕ' defined by $\Phi'(u_1, u_2) := u_1 + u_2 + G'(u_1, u_2)$ coincides with the application $\Phi(v_1, v_2) := v_1 + v_2 + G(v_1, v_2)$.

Due to Definition 3.1 the mapping Φ is a local diffeomorphism. So Φ' is also a local diffeomorphism. Thus G' satisfies Definition 3.1 of Hadamard graph. The first claim is proved.

The diffeomorphism $\Psi(v_1, v_2) = (u_1, u_2)$ as constructed above, converges to the identity map in the C^1 topology, when $\delta_1 \to 0^+$, and uniformly for all graphs G defined in $\mathbf{B}_\delta^{T_xM}(\mathbf{0})$. Thus, by Identity (16), $\|G' - G\|_{C^1}$ converges uniformly to zero, independently of the given graph G, when $\delta_1 \to 0$. This implies, in particular, that $\partial G'(u_1, u_2)/\partial u_2$ converges uniformly to $\partial G(v_1, v_2)/\partial v_2$ when $\delta_1 \to 0$. Thus, for any constant $c/2 > 0$ there exists $\delta_1 > 0$, which is independent of the graph G, such that $|\operatorname{disp}(G') - \operatorname{disp}(G)| < c/2$. In other words, $\operatorname{disp}(G') < c$ for all G with $\operatorname{disp}(G) < c/2$, as wanted. □

We are ready to prove the following Proposition, for all $f \in \operatorname{Diff}^1(M)$ with a dominated splitting $TM = E \oplus F$.

Proposition 3.7 *For all $\varepsilon > 0$ there are $\delta_0, K, n_0 > 0$, and a finite family of local foliations $\mathcal{L}$, each one defined in an open ball of a given finite covering of M with δ_0-balls, such that:*
(a) *$\mathcal{L}$ is C^1- trivializable and its leaves are $\dim F$-dimensional,*
(b) $\operatorname{dist}\left(F_{f^n(x)}, T_{f^n(x)} f^n(\mathcal{L}(x))\right) < \varepsilon$ for all x and for all $n \geq n_0$,
(c) *the following assertion holds for all $n \geq 0$ and for all x, y such that $y \in B_{\delta_0}^n(x)$:*

$$m^{f^n(\mathcal{L}(y))}(f^n(\mathcal{L}(y) \cap B_{\delta_0}^n(x))) \leq K,$$

(d) *the following inequality holds for all $n \geq 0$ and for all $x \in M$:*

$$e^{-n\varepsilon} K^{-1} \leq \frac{\left|\det df_x^n|_{T_x(\mathcal{L}(x))}\right|}{|\det df_x^n|_{F_x}|} \leq K e^{n\varepsilon}.$$

Proof: Consider the constant c' determined by Lemma 3.5. For each point $x \in M$ construct a local foliation $\mathcal{L}$ from a graph G defined on T_xM, with dispersion smaller than a constant $c/2$ such that $0 < c < c' < 1/2$. The constant c will be fixed later taking into account the given value of $\varepsilon > 0$.

After Lemma 3.6, there exists $\delta_1 > 0$ such that, for all $x \in M$ the graph G defined on $\mathbf{B}_\delta^{T_xM}(\mathbf{0})$ is redefined on $\mathbf{B}_\delta^{T_zM}(\mathbf{0})$, for any point $z \in B_{\delta_1}(x)$, preserving the same associated foliation and having dispersion upper bounded by c. Fix δ_0, n_0 (depending on c) by Lemma 3.5 and such that $\delta_0 < \delta_1$. For any given finite covering of M with balls $B_{\delta_0}(x_i)$, fix a finite family $\{\mathcal{L}_i\}_{1\leq i\leq k}$ of local foliations so constructed, one in each ball of the covering.

By the definition of graph, each foliation $\mathcal{L}$ of the finite family constructed above, is C^1-trivializable and its leaves have the same dimension as the dominating sub-bundle F. Thus Assertion (a) is proved.

From inequality (11), given $\varepsilon' > 0$ (a fixed value of $\varepsilon' > 0$ will be determined later), there exists $c > 0$ such that, if $\mathrm{disp}(G) < c$ then

$$\mathrm{dist}(T_x(\mathcal{L}(x), F_x)) < \varepsilon' \ \ \forall\ x \in M. \tag{19}$$

Recall that δ_0, n_0 (depending on c, which depends on ε') were defined by Lemma 3.5. Therefore, each leaf $f^n(\mathcal{L}(y) \bigcap B^n_{\delta_0}(x))$ is part of a single leaf of a foliation associated to a graph G_n, for all $n \geq 0$. Besides, Lemma 3.5 states that

$$\mathrm{disp}\, G_n < c \ \ \forall\, n \geq n_0 \tag{20}$$

From Inequalities (10) and (20) we deduce that

$$m^{f^n(\mathcal{L}(y))}(f^n(\mathcal{L}(y) \cap B^n_{\delta_0}(x))) \leq m^{f^n(\mathcal{L}(y))}(f^n(\mathcal{L}(y))) \leq$$

$$[(1 + \mathrm{disp}\, G_n)\, \delta\,]^{\ \dim F} < [(1 + c)\, \delta\,]^{\ \dim F} \quad \forall\ n \geq n_0.$$

Thus, there exists $K > 0$ such that

$$m^{f^n(\mathcal{L}(y))}(f^n(\mathcal{L}(y) \cap B^n_{\delta_0}(x))) \leq K \quad \forall\ n \geq 0.$$

So, Assertion (c) of Proposition 3.7 is proved for each fixed value of $\varepsilon' > 0$.

Next, we prove (d). From Inequalities (19) and (20), we deduce that

$$\mathrm{dist}\left(F_{f^n(x)}, T_{f^n(x)} f^n(\mathcal{L}(x))\right) < \varepsilon' \ \ \forall\ x \in M,\ \forall\ n \geq n_0. \tag{21}$$

Finally, we fix $\varepsilon' > 0$ (depending on the given value of $\varepsilon > 0$), such that $0 < \varepsilon' < \varepsilon$ and such that for all $\dim F$-dimensional sub-bundles L that satisfy $\mathrm{dist}(L, F) < \varepsilon'$, the following inequality holds:

$$e^{-\varepsilon} \leq \frac{|\det df_x|_{L(x)}|}{|\det df_x|_{F_x}|} \leq e^{\varepsilon} \ \ \forall\ x \in M. \tag{22}$$

Therefore, (21) implies:

$$e^{-\varepsilon} \leq \frac{\left|\det df_{f^j(x)}|_{T_{f^j(x)} f^j \mathcal{L}(x)}\right|}{\left|\det df_{f^j(x)}|_{F_{f^j(x)}}\right|} \leq e^{\varepsilon} \ \ \forall\ x \in M, \ \ \forall\ j \geq n_0.$$

The latter inequality implies (d). Finally, (b) is obtained from (21) taking into account that ε' was chosen smaller than ε. □

End of the proof of Lemma 2.2:

For the given value of $\varepsilon > 0$, we construct $\delta_0, K > 0$ as in Proposition 3.7. Consider any finite partition $\alpha = \{A_h\}_{1\le h\le k}$, where $k = \#(\alpha)$, such that $\operatorname{diam}\alpha = \max_{h=1\ldots k}\operatorname{diam}\{A_h\} < \delta_0$.

For each $A \in \alpha$ construct an open set $V_A \subset M$ also of diameter smaller than δ_0, containing A. Construct a $\dim F$ local foliation $\mathcal{L}_A$ in V_A satisfying Proposition 3.7. Construct also a C^1 submanifold W_A transversal to $\mathcal{L}_A$.

Take $\alpha^n = \vee_{j=0}^n f^{-j}(\alpha) = \{X_i\}_{1\le i\le k_n}$, where $k_n = \#(\alpha^n)$. For all $X_i \in \alpha^n$, there exists $A_{h_i} \in \alpha$ such that $X_i \subset A_{h_i}$. Denote $\mathcal{L}_i := \mathcal{L}_{A_{h_i}}$ and $W_i := W_{A_{h_i}}$. Since $\mathcal{L}_i$ is C^1-trivializable, by Fubini's Theorem we have:

$$m(C) = \sum_{i=1}^{k_n}\int_{z\in W_i} d\mu^{W_i}\int_{y\in\mathcal{L}_i(z)} \mathbf{1}_{C\bigcap X_i}\,\phi_i\, dm^{\mathcal{L}_i(z)} \qquad \forall\ C\in\mathcal{B}, \tag{23}$$

where $\mathcal{B}$ is the Borel sigma-algebra, $\mathbf{1}_{C\bigcap X_i}$ is the characteristic function of the set $C\bigcap X_i$, and ϕ_i is a continuous function which depends on $A_{h_i} \in \alpha$. Precisely, ϕ_i is the Jacobian of the C^1-trivialization of the foliation $\mathcal{L}_i$. So, there are at most $k = \#(\alpha)$ different local foliations $\mathcal{L}_i$, k different continuous functions ϕ_i, and k different transversal manifolds W_i, which allow Formula (23) work for any value of n and for any $C \in \mathcal{B}$.

Denote $\widehat{y} = f^n(y) \in f^n(\mathcal{L}_i(z)\bigcap X_i) =: \mathcal{L}_i^n(z)$:

$$m(C) = \sum_{i=1}^{k_n}\int_{z\in W_i} d\mu^{W_i}\int_{\widehat{y}\in\mathcal{L}_i^n(z)} [\mathbf{1}_{C\cap X_i}\ \phi_i](f^{-n}(\widehat{y}))\left|\det df^{-n}|_{T_{\widehat{y}}\mathcal{L}_i^n}\right| dm^{\mathcal{L}_i^n(z)}. \tag{24}$$

By Part (d) of Proposition 3.7:

$$\left|\det df^{-n}|_{T_{\widehat{y}}\mathcal{L}_i^n}\right| \le Ke^{n\varepsilon}\ \left|\det df^{-n}|_{F_{\widehat{y}}}\right|. \tag{25}$$

Recall Formula (3) defining $\psi_n(y)$. Since $f^n(y) = \widehat{y}$, we have

$$\log\left|\det df^{-n}|_{F_{\widehat{y}}}\right| = \psi_n(f^{-n}(\widehat{y})),$$

which together with inequality (25) and equality (24) gives:

$$\left|\det df^{-n}|_{T_{\widehat{y}}\mathcal{L}_i^n}\right| \le Ke^{n\varepsilon}e^{\psi_n(f^{-n}(\widehat{y}))} \tag{26}$$

$$m(C) \le Ke^{n\varepsilon}\sum_{i=1}^{k_n}\int_{z\in W_i} d\mu^{W_i}\int_{\widehat{y}\in\mathcal{L}_i^n(z)} [\mathbf{1}_{C\cap X_i}\ \phi_i](f^{-n}(\widehat{y}))\, e^{\psi_n(f^{-n}(\widehat{y}))}\, dm^{\mathcal{L}_i^n(z)}. \tag{27}$$

By Riesz Representation Theorem there exists a finite measure ν_n such that

$$\int h\, d\nu_n = \sum_{i=1}^{k_n}\int_{z\in W_i} d\mu^{W_i}\int_{\widehat{y}\in\mathcal{L}_i^n(z)} [(\mathbf{1}_{X_i}\cdot\phi_i\cdot h)\circ f^{-n}]\ (\widehat{y})\ dm^{\mathcal{L}_i^n(z)}\ \ \forall\, h\in C^0(M,\mathbb{R}).$$

From inequality (27) and the above definition of ν_n, we conclude

$$m(C) \leq Ke^{n\varepsilon} \int \mathbf{1}_C \, e^{\psi_n} \, d\nu_n \leq Ke^{n\varepsilon} \int_C e^{\psi_n} \, d\nu_n.$$

Statement (ii) of Lemma 2.2 is proved.

Let us prove Statement (i). We must show that there exists a constant $K_0 > 0$, independent of n, such that $\nu_n(X) \leq K_0$ for all $X \in \alpha^n$, and for all $n \geq 0$. In fact, recall that $\mathcal{L}_i^n(z) = f^n(\mathcal{L}_i(z) \cap X_i) \subset f^n(\mathcal{L}_i(z) \cap B_{\delta_0}(y))$ for all $z \in W_i$ and for all $y \in \mathcal{L}_i(z) \cap X_i$. Thus, applying Property (c) of Proposition 3.7 we have

$$m^{\mathcal{L}_i^n}(\mathcal{L}_i^n(z)) \leq K_1,$$

for some constant $K_1 > 0$ which is independent of n. From the construction of the measure ν_n:

$$\nu_n(X_i) = \int_{z \in W_i} d\mu^{W_i} \int_{\widehat{y} \in \mathcal{L}_i^n(z)} \left[(\mathbf{1}_{X_i} \cdot \phi_i) \circ f^{-n} \right](\widehat{y}) \, dm^{\mathcal{L}_i^n(z)} \leq$$

$$\mu^{W_i}(W_i) \, \|\phi_i\|_{C^0} \; m^{\mathcal{L}_i^n}(\mathcal{L}_i^n(z)) \leq K_1 \, \mu^{W_i}(W_i) \, \|\phi_i\|_{C^0}.$$

Since the number of different local foliations $\mathcal{L}_i$ is equal to the number k of pieces of the given partition α, which is independent of n, we obtain:

$$\nu_n(X_i) \leq K_1 \max_{A \in \alpha} \{\mu^{W_A}(W_A) \, \|\phi_A\|_{C^0}\} =: K_0,$$

where K_0 depends only on the partition α and not on n. □

4 Proof of Lemma 2.3

Choose $\{\varphi_i\}_{i \geq 1}$ dense in $C^0(M, [0,1])$ and define dist^* in $\mathcal{P}$:

$$\text{dist}^*(\mu_1, \mu_2) := \left| \int \psi \, d\mu_1 - \int \psi \, d\mu_2 \right| + \sum_{i=1}^{\infty} \frac{\left| \int \varphi_i \, d\mu_1 - \int \varphi_i \, d\mu_2 \right|}{2^i}. \tag{28}$$

By hypothesis, a measure $\mu \in \mathcal{P}_f$ and two small numbers $\varepsilon > 0$ and $\delta > 0$ are arbitrarily given. We must construct an adequate finite partition α of M, with diameter smaller than δ, satisfying Lemma 2.3.

Take $\delta_1 > 0$ such that $\text{dist}(x, y) < \delta_1 \Rightarrow |\psi(x) - \psi(y)| < \varepsilon/5$.
Take α such that $\text{diam}(\alpha) \leq \min(\delta, \delta_1)$, $\mu(\partial X) = 0 \; \forall \; X \in \alpha$. This construction implies

$$\lim_{n \to +\infty} \mu_n(X) = \mu(X) \tag{29}$$

for all $X \in \alpha^q = \bigvee_{j=0}^{q} f^{-j}(\alpha)$, for all $q \in \mathbb{N}$ and for all $\{\mu_n\}_n \subset \mathcal{P}$ such that $\lim_n^* \mu_n = \mu$. Also

$$|\psi_n(y) - \psi_n(x)| \leq \sum_{j=0}^{n-1} |\psi(f^j(y)) - \psi(f^j(x))| \leq \frac{n\varepsilon}{5} \;\; \forall \; x, y \in X, \forall X \in \alpha^n. \tag{30}$$

Recall that $h_\mu(\alpha) := \lim_{q\to+\infty} H(\alpha^q,\mu)/q$, where

$$H(\alpha^q,\mu) := -\sum_{X\in\alpha^q} \mu(X)\log\mu(X).$$

Fix $q\in\mathbb{N}^+$ such that $H(\alpha^q,\mu)/q < h_\mu(\alpha)+\varepsilon/5$. From (29):

$$\lim_{n\to+\infty}\frac{H(\alpha^q,\mu_n)}{q} = \frac{H(\alpha^q,\mu)}{q} < h_\mu(\alpha)+\frac{\varepsilon}{5} \tag{31}$$

for any sequence $\mu_n\in\mathcal{P}$ such that $\lim_n^* \mu_n = \mu$.

Using (31), fix $0<\varepsilon_0^*<\varepsilon/5$ such that

$$\sigma\in\mathcal{P},\ \mathrm{dist}^*(\sigma,\mu)\le\varepsilon_0^* \Rightarrow |H(\alpha^q,\sigma)-H(\alpha^q,\mu)|\le\frac{q\varepsilon}{5}. \tag{32}$$

Such a value of ε_0^* exists; otherwise we could construct a sequence of probability measures μ_n converging to μ and such that $|H(\alpha^q,\mu_n)-H(\alpha^q,\mu)|>q\,\varepsilon/5$ for all $n\in\mathbb{N}$. This inequality contradicts the equality at left in (31).

For any fixed $0<\varepsilon^*<\varepsilon_0^*$ we denote $C_n = C_n(\varepsilon^*)$ defined by equality (4). Consider

$$\alpha^n\bigvee\{C_n\} := \{X_i\cap C_n:\quad X_i\in\alpha^n,\quad X_i\cap C_n\neq\emptyset\}.$$

Denote $k_n := \#(\alpha\bigvee\{C_n\})$. For each $C_n\cap X_i\in\alpha^n\bigvee\{C_n\}$, choose one point $x_i\in C_n\bigcap X_i$. Consider the integral $I(\psi_n,C_n,\nu_n)$ defined by equality (6), and apply inequality (30):

$$I_n := I(\psi_n,C_n,\nu_n) = \int_{C_n} e^{\psi_n}\,d\nu_n = \sum_{i=1}^{k_n}\int_{y\in C_n\cap X_i} e^{\psi_n(y)}d\nu_n(y) \le$$

$$\sum_{i=1}^{k_n} e^{n\varepsilon/5}e^{\psi_n(x_i)}\nu_n(C_n\cap X_i). \tag{33}$$

By hypothesis $\nu_n(X)\le K$ for all $X\in\alpha^n$. So:

$$I_n \le K e^{n\varepsilon/5}\sum_{i=1}^{k_n} e^{\psi_n(x_i)} \tag{34}$$

Define $p_i := e^{\psi_n(x_i)}/L$, where $L := \sum_{i=1}^{k_n} e^{\psi_n(x_i)}$. Note that $\sum_{i=1}^{k_n} p_i = 1$. Then:

$$\log\sum_{i=1}^{k_n} e^{\psi_n(x_i)} = \sum_{i=1}^{k_n}\psi_n(x_i)p_i - \sum_{i=1}^{k_n} p_i\log p_i. \tag{35}$$

Taking logarithm in (34) and using (35), we obtain:

$$\log I_n \le \log K + \frac{n\varepsilon}{5} + \sum_{i=1}^{k_n}\psi_n(x_i)p_i - \sum_{i=1}^{k_n} p_i\log p_i.$$

From Equalities (2) and (3):

$$\sum_{i=1}^{k_n} \psi_n(x_i) p_i = \sum_{i=1}^{k_n} \sum_{j=0}^{n-1} \int p_i \, \psi \; d\delta_{f^j(x_i)},$$

and thus:

$$\log I_n \leq \log K + \frac{n\varepsilon}{5} + \sum_{i=1}^{k_n} \sum_{j=0}^{n-1} \int p_i \, \psi \; d\delta_{f^j(x_i)} - \sum_{i=1}^{k_n} p_i \log p_i. \tag{36}$$

Let $\sigma_{n,x}$ be the empirical probability according to Definition 1.2. We construct $\mu_n \in \mathcal{P}$ by the following equality:

$$\mu_n := \frac{1}{n} \sum_{i=1}^{k_n} \sum_{j=0}^{n-1} p_i \; \delta_{f^j(x_i)} = \sum_{i=1}^{k_n} p_i \sigma_{n,x_i}. \tag{37}$$

Since $x_i \in C_n$ we have that $\text{dist}^*(\sigma_{n,x_i}, \mu) < \varepsilon^*$ - see equality (4). Since the ε^*-balls defined with the metric dist^* by equality (28) are convex, and μ_n is a convex combination of the measures σ_{n,x_i}, we deduce

$$\text{dist}^*(\sigma_{n,x_i}, \mu) \leq \varepsilon^* \Rightarrow \text{dist}^*(\mu_n, \mu) \leq \varepsilon^*.$$

From the construction of dist^* by equality (28), we obtain $\left| \int \psi \, d\mu_n - \int \psi \, d\mu \right| \leq \varepsilon^* < \varepsilon/5$. Therefore:

$$\int \psi \, d\mu_n \leq \int \psi \, d\mu + \frac{\varepsilon}{5},$$

which togethter with (36) and (37) implies:

$$\log I_n \leq \log K + \frac{2n\varepsilon}{5} + n \int \psi \, d\mu - \sum_{i=1}^{k_n} p_i \log p_i.$$

In Assertion 5.3 of the appendix we prove the following statement:

There exists $n_0 \geq 0$ such that

$$-\sum_{i=1}^{k_n} p_i \log p_i \leq \frac{n\,\varepsilon}{5} + \frac{n}{q} H(\alpha^q, \mu_n) \quad \forall n \geq n_0$$

Therefore:

$$\log I_n \leq \log K + \frac{3n\varepsilon}{5} + n \int \psi \, d\mu + \frac{nH(\alpha^q, \mu_n)}{q}.$$

By the construction of ε_0^* in (32), and since $\text{dist}^*(\mu_n, \mu) < \varepsilon^* < \varepsilon_0^*$, we deduce:

$$\left| \frac{H(\alpha^q, \mu_n)}{q} - \frac{H(\alpha^q, \mu)}{q} \right| \leq \frac{\varepsilon}{5}.$$

So

$$\log I_n \leq \log K + \frac{4n\varepsilon}{5} + n \int \psi \, d\mu + \frac{n\, H(\alpha^q, \mu)}{q}.$$

Finally, using the choice of q by inequality (31) we conclude

$$\log I_n \leq \log K + n\,\varepsilon + n \int \psi \, d\mu + n\, h_\mu(\alpha),$$

ending the proof of Lemma 2.3. □

5 Appendix

In this appendix we check some technical assertions that were used in the proofs of Sections 3 and 4.

Assertion 5.1 *Let $\delta > 0$ be such that for all $x \in M$*

$$\exp_x \colon \{v \in T_xM : \|v\| \leq 3\delta\} \to B_{3\delta}(x) \subset M$$

is a diffeomorphism. Let $n_0 > 0$.

Then, there exists $0 < \delta_0 < \delta$ such that for all $x \in M$, for all $0 \leq n \leq n_0$ and for any graph G (defined in $\mathbf{B}_\delta^{T_xM}(\mathbf{0}) \subset T_xM$) *with*

$$\operatorname{disp} G < 1/2,$$

there exists a graph G_n (defined in $\mathbf{B}_\delta^{T_{f^n(x)}M}(\mathbf{0})$) *satisfying the following condition:*

$$\begin{array}{l} \text{for all } y = \exp_x(v_1 + v_2 + G(v_1, v_2)) \in B_{\delta_0}(x) \\ \text{there exists } (u_1, u_2) \in \mathbf{B}_\delta^{E_{f^n(x)}}(\mathbf{0}) \times \mathbf{B}_\delta^{F_{f^n(x)}}(\mathbf{0}) \\ \text{where } u_1 \text{ depends only on } v_1 \\ \text{and } f^n(y) = \exp_{f^n(x)}(u_1 + u_2 + G_n(u_1, u_2)). \end{array} \tag{38}$$

Proof: We will argue by induction on $n \in \mathbb{N}$, to show that for each $n \geq 1$, there exists $\delta_n > 0$ and G_n satisfying statement (38). To prove Assertion 5.1 it is enough to take $\delta_0 := \min\{\delta_1, \ldots, \delta_{n_0}\}$.

To simplify the notation along the proof, we will not write the exponential maps. From Definition 3.1, recall the construction of the diffeomorphism Φ obtained from the graph G, which is a trivialization of the associated local foliation $\mathcal{L}$ (see the upper frame of Figure 2). Precisely, each leaf $\mathcal{L}(v_1)$ is obtained for constant $v_1 \in \mathbf{B}_\delta^{E_x}(\mathbf{0})$, and parametrized by $v_2 \in \mathbf{B}_\delta^{F_x}(\mathbf{0})$ through the formula

$$\mathcal{L}(v_1) : v_2 \mapsto \Phi(v_1, v_2) := v_1 + v_2 + G(v_1, v_2), \quad \text{where } G(v_1, v_2) \in \mathbf{B}_\delta^{E_x}(\mathbf{0}).$$

Since $G(v_1, 0) = 0$ we have $v_1 = \Phi(v_1, 0)$. i.e.

$$v_1 \in \mathcal{L}(v_1) \cap \mathbf{B}_\delta^{E_x}(\mathbf{0}).$$

Moreover,

$$\pi_{F_x}(v_1 + v_2 + G(v_1, v_2)) = v_2 \text{ for all } v_2 \in \mathbf{B}_\delta^{F_x}(\mathbf{0}).$$

So

$$\pi_{F_x}\mathcal{L}(v_1) = \mathbf{B}_\delta^{F_x}(\mathbf{0}).$$

Besides, $\mathcal{L}(v_1)$ is uniformly transversal to E_x, for all G with $\operatorname{disp}(G) < 1/2$. In fact

$$T_{v_1}\mathcal{L}(v_1) = \operatorname{Im}(\mathrm{Id}|_{F_x} + \partial G/\partial v_2),$$

the subspace F_x is transversal to E_x, and

$$\|\partial G/\partial v_2\| \leq \operatorname{disp} G < 1/2.$$

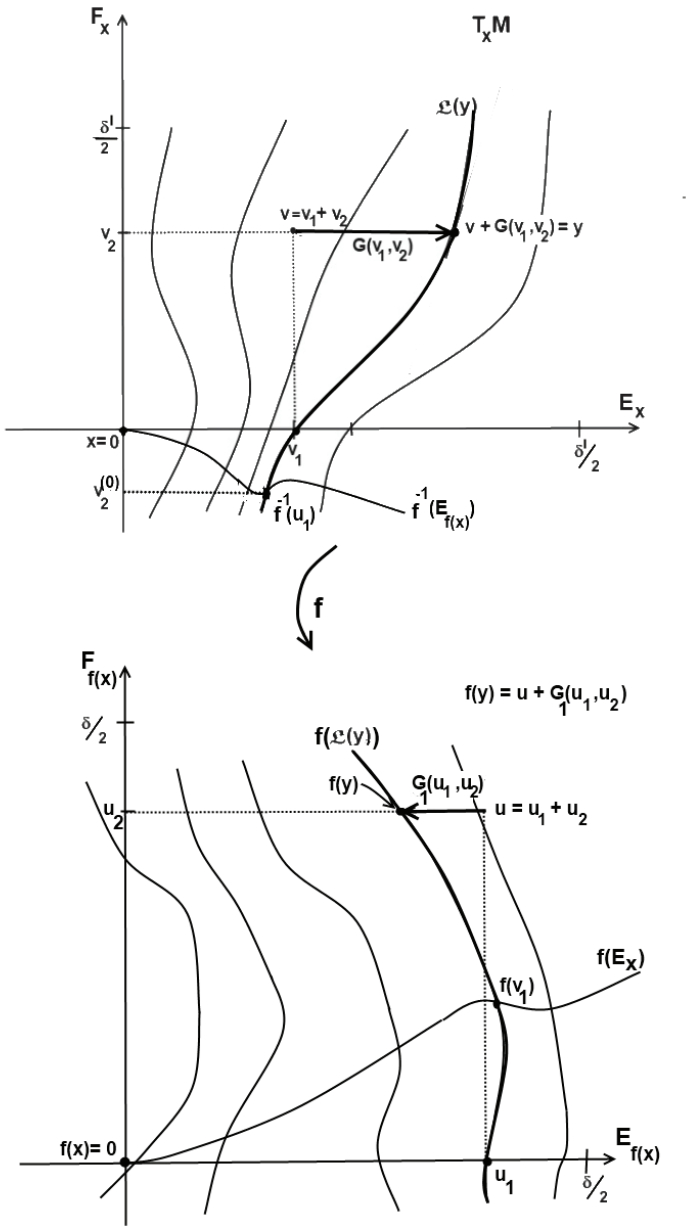

Figure 2: The image $f(\mathcal{L}(y))$ of the leaves $\mathcal{L}(y)$ near x associated to the graph G, are associated to the graph G_1. (We omit the exponential maps $\exp_x$, $\exp_{f(x)}$).

Thus, since the leaf $\mathcal{L}(x)$ intersects E_x at 0, we deduce that there exists $0 < \delta' \leq \delta$, which is uniform for any G with $\mathrm{disp}(G) < 1/2$, such that, if $\mathrm{dist}(x, y) < \delta'$ then y' belongs to some leaf of the foliation $\mathcal{L}$. In other words,

$$B_{\delta'}(x) \subset \mathrm{Im}(\Phi),$$

i.e. there exists $(v_1, v_2) \in E_x \times F_x$ such that

$$\|v_1\| < \delta, \quad \|v_2\| < \delta \text{ and}$$

$$y = \Phi(v_1, v_2) = v_1 + v_2 + G(v_1, v_2) \quad \text{if } y \in B_{\delta'}(x).$$

Recall that, by Definition 3.1, Φ is a diffeomorphism onto its image. Thus, for all $y \in B_{\delta'}(x)$, the point $\Phi^{-1}(y) = (v_1, v_2) \in \mathbf{B}_\delta^{E_x}(\mathbf{0}) \times \mathbf{B}_\delta^{F_x}(\mathbf{0})$ depends C^1 on y. We take $0 < \delta_1 < \delta'$ such that if $y \in B_{\delta_1}(x)$, then

$$\|\pi_{E_{f(x)}} f(y)\| < \delta/2, \quad \|\pi_{F_{f(x)}} f(y)\| < \delta/2. \tag{39}$$

Such a value of $\delta_1 > 0$ exists, and is independent of the graph G, because f, π_{E_x} and π_{F_x} are uniformly continuous.

Taking if necessary a smaller value of δ_1, the following two properties (A) and (B) hold for any graph G with $\mathrm{disp}(G) < 1/2$ and for any $y \in B_{\delta_1}(x)$:

(A) The leaf $f(\mathcal{L}(y))$ intersects $\mathbf{B}_{\delta/2}^{E_{f(x)}}(\mathbf{0}) \subset E_{f(x)}$ in a point u_1 (see Figure 2). In other words

$$\text{there exists} \quad u_1 \in E_{f(x)}, \quad \|u_1\| < \delta/2, \quad f^{-1}(u_1) \in \mathcal{L}(y). \tag{40}$$

(B) The application $v_1 \in E_x \mapsto u_1 \in E_{f(x)}$ defined by (A) for all $y = \Phi(v_1, v_2) \in B_{\delta_0}(x)$, is independent of v_2, and is a diffeomorphism onto its image.

Property (A) is achieved due to the Implicit Function Theorem, since f is a diffeomorphism, $\mathrm{Im}\, df_x|_{E_x} = E_{f(x)}$, and the local foliation $\mathcal{L}$ is uniformly transversal to $\mathbf{B}_\delta^{E_x}(\mathbf{0}) \subset E_x$, while its leaf $\mathcal{L}(x)$ intersects E_x at 0. Property (B) is obtained because f is a diffeomorphism and the mapping $f(v_1) \in f(E_x) \mapsto u_1 \in E_{f(x)}$ is the holonomy along the leaves of the foliation $f(\mathcal{L})$, which is C^1 trivializable and uniformly transversal to both $f(E_x)$ and $E_{f(x)}$. (See Figure 2.)

Let us show that the graph G_1 exists in $T_{f(x)}M$ satisfying Definition 3.1 and Assertion (38) for all $y \in B_{\delta_1}(x)$. We write $y = \Phi(v_1, v_2) = v_1 + v_2 + G(v_1, v_2)$. We have already determined $u_1 \in E_{f(x)}$ as a diffeomorphic function of v_1, which does not depend on v_2. Let us determine $u_2 \in F_{f(x)}$ and $G_1(u_1, u_2) \in E_{f(x)}$ such that $f(y) = u_1 + u_2 + G_1(u_1, u_2)$ according to Figure 2. Consider the equation:

$$f(v_1 + v_2 + G(v_1, v_2)) = u_1 + u_2 + G_1(u_1, u_2), \tag{41}$$

$$\text{where} \quad u_2 \in F_{f(x)} \quad \text{and} \quad G_1(u_1, u_2) \in E_{f(x)}.$$

Equation (41) is solved by

$$u_2 := \pi_{F_{f(x)}} f(v_1 + v_2 + G(v_1, v_2)) \in F_{f(x)}, \tag{42}$$

$$G_1(u_1, u_2) := -u_1 + \pi_{E_{f(x)}} f(v_1 + v_2 + G(v_1, v_2)) \in E_{f(x)}. \tag{43}$$

The application Ψ defined by $\Psi(v_1, v_2) = (u_1, u_2)$, where u_1 and u_2 are constructed as above, is of C^1 class. In fact, u_1 depends only on v_1, the mapping $v_1 \mapsto u_1$ is a diffeomorphism onto its image, and u_2 is constructed by Formula (42).

Moreover, Ψ is C^1 invertible. In fact, on the one hand, we have that the application $v_1 \mapsto u_1$ is C^1 invertible and independent of v_2. On the other hand, for constant v_1 let us show that Formula (42) applies $v_2 \mapsto u_2$ C^1-diffeomorphically. Precisely, $G(v_1, 0) = 0$, $v_2 \in F_x$ $u_2 \in F_{f(x)}$, $G(v_1, v_2) \in E_x$ and $df_x \colon F_x \to F_{f(x)}$ is invertible. Thus, $\pi|_{F_{f(x)}}\, df = df|_{F_x}\, \pi_{F_x}$. Taking derivatives in equality (42) with respect to v_2 with constant v_1, we obtain:

$$\frac{\partial u_2}{\partial v_2} = df|_{F_x} \cdot \pi_{F_x} \left(Id|_{F_x} + \frac{\partial G(v_1, v_2)}{\partial v_2} \right) = df|_{F_x} = (df^{-1}|_{F_{f(x)}})^{-1}.$$

The second equality holds because $G(v_1, v_2) \in E_x$ for all (v_1, v_2), and so, the projection by π_{F_x} composed with any derivative of G, equals zero. We have proved that $\partial u_2 / \partial v_2$ is invertible, and besides

$$\left(\frac{\partial u_2}{\partial v_2} \right)^{-1} = \frac{\partial v_2}{\partial u_2} = df^{-1}|_{F_{f(x)}} \tag{44}$$

concluding that the application Ψ is a diffeomorphism onto its image.

Now, we define the mapping Φ_1 by

$$\Phi_1(u_1, u_2) = u_1 + u_2 + G_1(u_1, u_2).$$

Φ_1 is a C^1 diffeomorphism onto its image, because its inverse is $\Psi \circ \Phi \circ f^{-1}$. So G_1 is C^1, and Φ_1 is the C^1 trivialization of its associated foliation, which is, by construction, $f(\mathcal{L})$.

Finally, (39) , (40) and (43) imply

$$\|G_1\| \leq \|u_1\| + \|\pi_{E_{f(x)}} f(y)\| < \delta/2 + \delta/2 = \delta \quad \text{and}$$

$$\Phi_1^{-1} f(B_{\delta_1}(x)) \subset \mathbf{B}_{\delta/2}^{E_{f(x)}}(\mathbf{0}) \times \mathbf{B}_{\delta/2}^{F_{f(x)}}(\mathbf{0}).$$

Thus, $G_1 \colon \Phi_1^{-1}(f(B_{\delta_1}(y))) \to \mathbf{B}_{\delta}^{E_{f(x)}}(\mathbf{0})$ can be C^1 extended to be a graph $G_1 \colon \mathbf{B}_{\delta}^{E_{f(x)}}(\mathbf{0}) \times \mathbf{B}_{\delta}^{F_{f(x)}}(\mathbf{0}) \to \mathbf{B}_{\delta}^{E_{f(x)}}(\mathbf{0})$.

We have completed the first step of the inductive proof, since we have proved the existence of $\delta_1 > 0$ and of the graph G_1 satisfying (38). Naturally, disp G_1 is not necessarily upper bounded by $1/2$. So, we can not exactly repeat the same argument to prove the inductive step. We will instead prove that there exists a uniform constant $c_1 > 0$ such that

$$\text{if } \operatorname{disp} G < 1/2 \text{ then } \operatorname{disp} G_1 < c_1. \tag{45}$$

If we prove inequality (45) for some constant c_1, then we can end the inductive proof as follows.

Assume that for some $n \geq 0$ there are $\delta_n, c_n > 0$ and a graph G_n defined in $\mathbf{B}_\delta^{T_{f^n(x)}M}(\mathbf{0})$ satisfying (38) for all $y \in B_{\delta_n}(x)$, and such that $\operatorname{disp} G_n \leq c_n$ for any graph G with $\operatorname{disp} G < 1/2$. Thus, we can repeat the above proof, putting $\min(\delta_n, \delta), c_n$ and G_n in the roles of $\delta, 1/2$ and G respectively. We deduce that there are $\delta_{n+1}, c_{n+1} > 0$ and a graph $G_{n+1} := (G_n)_1$, defined in $\mathbf{B}_\delta^{T_{f^{n+1}(x)}M}(\mathbf{0})$, which satisfies (38) for all $y \in B_{\delta_{n+1}}(x)$, and such that $\operatorname{disp} G_{n+1} < c_{n+1}$ for any graph G for which $\operatorname{disp} G_n < c_n$. Thus, G_{n+1} satisfies (38) for all G such that $\operatorname{disp} G < 1/2$. Therefore, the inductive proof will be completed once we show inequality (45).

So, let us find a constant c_1 satisfying inequality (45). To find c_1 we will bound from above the term $\|\partial G_1(u_1, u_2)/\partial u_2\|$. From (43), and taking into account that

$$\pi_{E_{f(x)}} \cdot df|_x = df|_{E_x} \cdot \pi|_{E_x},$$

we obtain:

$$\frac{\partial G_1(u_1,u_2)}{\partial u_2} = df|_{E_x} \cdot \pi|_{E_x} \cdot \left(Id|_{F_x} + \frac{\partial G(v_1,v_2)}{\partial v_2} \right) \cdot \frac{\partial v_2}{\partial u_2} = df|_{E_x} \cdot \frac{\partial G(v_1,v_2)}{\partial v_2} \cdot \frac{\partial v_2}{\partial u_2}.$$

Applying (44) and the definition of dispersion, we deduce

$$\operatorname{disp}(G_1) \leq \|df|_{E_x}\| \cdot \operatorname{disp}(G) \cdot \|df^{-1}|_{F_{f(x)}}\|. \tag{46}$$

Thus, inequality (45) follows taking

$$c_1 := \max_{x \in M} \{\|df|_{E_x}\| \cdot \|df^{-1}|_{F_{f(x)}}\|\},$$

ending the proof of Assertion 5.1. □

Assertion 5.2 *Let $\delta > 0$ be such that for all $x \in M$*

$$\exp_x \colon \{v \in T_xM : \|v\| \leq 3\delta\} \to B_{3\delta}(x) \subset M$$

is a diffeomorphism. For all $0 < \delta_0 < \delta$ there exists $0 < c' < 1/2$ satisfying the following property:

Assume that G is a Hadamard graph defined in $\mathbf{B}_\delta^{T_xM}(\mathbf{0})$ such that

$$\operatorname{disp} G < c'.$$

Assume that there exists $n \in \mathbb{N}$ and a graph G_n in $\mathbf{B}_\delta^{T_{f^n(x)}M}(\mathbf{0}) \subset T_{f^n(x)}M$ such that

$$\begin{array}{l} \text{for all } y = \exp_x(v_1 + v_2 + G(v_1, v_2)) \in B^n_{\delta_0}(x) \\ \text{there exists } (u_1, u_2) \in \mathbf{B}_\delta^{E_{f^n(x)}}(\mathbf{0}) \times \mathbf{B}_\delta^{F_{f^n(x)}}(\mathbf{0}), \\ \text{where } u_1 \text{ depends only on } v_1 \text{ and} \\ f^n(y) = \exp_{f^n(x)}(u_1 + u_2 + G_n(u_1, u_2)). \end{array} \tag{47}$$

Then, the following inequality holds for all $y = \exp_x(v_1 + v_2 + G(v_1, v_2)) \in B^n_{\delta_0}(x)$:

$$\left\| \frac{\partial G_n}{\partial u_2}(u_1, u_2) \right\| \leq \left\| df^n|_{E_x} \right\| \cdot \operatorname{disp} G \cdot \left\| df^{-n}|_{F_{f^n(x)}} \right\|. \tag{48}$$

Proof: To simplify the notation, we do not write the exponential maps.

Equality (47) can be written as follows:

$$f^n(v_1 + v_2 + G(v_1, v_2)) = u_1 + u_2 + G_n(u_1, u_2), \tag{49}$$

where

$$(v_1, v_2) \in E_x \times F_x, \quad (u_1, u_2) \in E_{f^n(x)} \times F_{f^n(x)},$$
$$G \in E_x, \quad G_n \in E_{f^n(x)} \quad \text{and}$$
$$y = v_1 + v_2 + G(v_1, v_2) \in B^n_{\delta_0}(x).$$

Then:

$$u_2 = \pi_{F_{f^n(x)}} f^n(v_1 + v_2 + G(v_1, v_2)), \tag{50}$$

$$G_n(u_1, u_2) = -u_1 + \pi_{E_{f^n(x)}} f^n(v_1 + v_2 + G(v_1, v_2)). \tag{51}$$

Taking derivatives in equality (50) with respect to v_2, with constant v_1, and noting that $\pi|_{F_{f^n(x)}} \cdot df = df|_{F_x} \cdot \pi_{F_x}$, we obtain:

$$\frac{\partial u_2}{\partial v_2} = df^n|_{F_x} \pi_{F_x} (Id|_{F_x} + (\partial G/\partial v_2)) = df^n|_{F_x} = (df^{-n}|_{F_{f^n(x)}})^{-1}.$$

In the second equality above, we used that $G(v_1, v_2) \in E_x$ for all (v_1, v_2) (recall Definition 3.1 of Hadamard graphs). Since $df^n|_{F_x}$ is invertible, the linear transformation $\partial u_2/\partial v_2$ is also invertible, and

$$\left(\frac{\partial u_2}{\partial v_2}\right)^{-1} = \frac{\partial v_2}{\partial u_2} = df^{-n}|_{F_{f^n(x)}}.$$

Now, we take derivatives in equality (51) with respect to v_2 with constant v_1. We recall that, by hypothesis, u_1 depends only on v_1, but not on v_2. Besides, we note that $\pi|_{E_{f^n(x)}} \cdot df^n = df^n|_{E_x} \cdot \pi|_{E^x}$. Thus:

$$\frac{\partial G_n}{\partial u_2} \cdot \frac{\partial u_2}{\partial v_2} = df^n|_{E_x} \cdot \frac{\partial G}{\partial v_2}.$$

Thus:

$$\frac{\partial G_n}{\partial u_2} = df^n|_{E_x} \cdot \frac{\partial G}{\partial v_2} \cdot \frac{\partial v_2}{\partial u_2} = df^n|_{E_x} \cdot \frac{\partial G}{\partial v_2} \cdot df^{-n}|_{F_{f^n(x)}}.$$

So, after Definition 3.2 of $\mathrm{disp}(G)$ we deduce

$$\left\| \frac{\partial G_n}{\partial u_2} \right\| \leq \|df^n|_{E_x}\| \cdot \mathrm{disp}\, G \cdot \left\| df^{-n}|_{F_{f^n(x)}} \right\|,$$

proving inequality (48). □

Assertion 5.3 *There exists $n_0 \geq 1$ such that*

$$-\sum_{i=1}^{k_n} p_i \log p_i \leq \frac{n\varepsilon}{5} + \frac{nH(\alpha^q, \mu_n)}{q} \qquad \forall\ n \geq n_0,$$

where $\varepsilon > 0$, α is a finite partition, $\alpha^n = \bigvee_{j=0}^{n} f^{-j}(\alpha)$, with $k_n \leq \#(\alpha^n)$, and

$$0 \leq p_i \leq 1, \qquad \sum_{i=1}^{k_n} p_i = 1, \qquad \mu_n := \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=1}^{k_n} p_i \delta_{f^j(x_i)}$$

with $x_i \in X_i$, $X_i \in \alpha^n$ and

$$H(\alpha^q, \mu_n) := - \sum_{A \in \alpha^q} \mu_n(A) \log \mu_n(A).$$

Proof: Denote $k := \#\alpha$. Construct the probability measure $\pi_n := \sum_{i=1}^{k_n} p_i \delta_{x_i}$. Then $\pi_n(X_i) = p_i \ \ \forall\ 1 \leq i \leq k_n$ and

$$H(\alpha^n, \pi_n) = - \sum_{i=1}^{k_n} p_i \log p_i$$

Fix $0 \leq l \leq q-1$. Since α^{n+l} is thinner than α^n, we have $H(\alpha^n, \pi_n) \leq H(\alpha^{n+l}, \pi_n)$. Thus

$$- \sum_{i=1}^{k_n} p_i \log p_i \leq H(\alpha^{n+l}, \pi_n), \tag{52}$$

where

$$\alpha^{n+l} = \vee_{j=0}^{n+l} f^{-j}\alpha = \Big(\vee_{j=0}^{l-1} f^{-j}\alpha \Big) \vee \Big(f^{-l}(\vee_{j=0}^{n} f^{-j}\alpha) \Big).$$

Besides, for any two partitions α and β, and for any probability measure ν we have $H(\alpha \vee \beta, \nu) \leq H(\alpha, \nu) + H(\beta, \nu)$. Therefore

$$H(\alpha^{n+l}, \pi_n) \leq \sum_{j=0}^{l-1} H(\alpha, f^{*j}\pi_n) + H(f^{-l}\alpha^n, \pi_n), \tag{53}$$

where the operator $f^*\colon \mathcal{P} \to \mathcal{P}$ in the space of probability measures is defined by $f^*(\nu)(B) = \nu(f^{-1}(B))$ for any measurable set B.
Since $H(\alpha, \nu) \leq \log(\#(\alpha)) = \log k$ for any probability measure ν, and since $0 \leq l < q$, from Inequalities (52) and (53), we obtain:

$$- \sum_{i=1}^{k_n} p_i \log p_i \leq q \log k + H(\alpha^n, f^{*l}\pi_n).$$

If $n \geq (10\, q\, \log k) \,/\, \varepsilon$, then

$$- \sum_{i=1}^{k_n} p_i \log p_i \leq \frac{n\varepsilon}{10} + H(\alpha^n, f^{*l}\pi_n). \tag{54}$$

Now we write: $n = Nq + s,\ 0 \leq s \leq q-1$. We have

$$H(\alpha^n, f^{*l}\pi_n) \leq \sum_{h=0}^{N-1} H(\alpha^q, f^{*hq+l}\pi_n) + \sum_{j=Nq}^{Nq+s} H(\alpha, f^{*j+l}\pi_n)$$

Arguing as above:

$$\sum_{j=Nq}^{Nq+s} H(\alpha, f^{*j+l}\pi_n) \leq (s+1)\log k \leq q \log k \leq \frac{n\varepsilon}{10}.$$

So, inequality (54) implies:

$$-\sum_{i=1}^{k_n} p_i \log p_i \leq \frac{n\varepsilon}{5} + \sum_{h=0}^{N-1} H(\alpha^q, f^{*hq+l}\pi_n)$$

Taking all values of l such that $0 \leq l \leq q-1$ and adding the above bounds, we deduce:

$$-q\sum_{i=1}^{k_n} p_i \log p_i \leq \frac{nq\varepsilon}{5} + \sum_{h=0}^{N-1}\sum_{l=0}^{q-1} H(\alpha^q, f^{*hq+l}\pi_n) \leq \frac{nq\varepsilon}{5} + \sum_{j=0}^{n-1} H(\alpha^q, f^{*j}\pi_n). \quad (55)$$

Recall that the entropy H of a partition with respect to a convex combination of probabilities, is not smaller than the convex combination of the entropies with respect to each of the probabilities. Since $\mu_n = \frac{1}{n}\sum_{j=1}^{n-1}\sum_{i=1}^{k_n} p_i \delta_{f^j(x_i)} = \frac{1}{n}\sum_{j=0}^{n-1} f^{*j}\pi_n$, we deduce $\frac{1}{n}\sum_{j=0}^{n-1} H(\alpha^q, f^{*j}\pi_n) \leq H(\alpha^q, \mu_n)$. Substituting in inequality (55) we conclude

$$-q\sum_{i=1}^{k_n} p_i \log p_i \leq \frac{nq\varepsilon}{5} + nH(\alpha^q, \mu_n),$$

ending the proof of Assertion 5.3. □

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References

[ABC11] Abdenur, F.; Bonatti, C.; Crovisier, S.: *Nonuniform hyperbolicity for C^1-generic diffeomorphisms.* Israel J. Math. **183** 2011 pp. 1-60.

[AB06] Avila, A.; Bochi, J.: *A generic C^1 map has no absolutely continuous invariant probability measure.* Nonlinearity **19** 2006 pp. 2717-2725.

[BP07] Barreira, L.; Pesin, Y.: *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents.* Encyclopedia of Mathematics and its Applications, Cambridge University Press 2007.

[BCS13] Bonatti,C.; Crovisier, S; Shinohara K.: *The $C^{1+\alpha}$ hypothesis in Pesin theory revisited.* Preprint arXiv 1306.6391v1[math.DS]

[BDV05] Bonatti, C.; Díaz, L.; Viana, M.: *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective.* Springer, Encyclopaedia of Mathematical Sciences **102.** 2005. ISBN 3-540-22066-6.

[CE11] Catsigeras, E.; Enrich, H.: *SRB-like measures for C^0 dynamics.* Bull. Pol. Acad. Sci. Math. **59** 2011 pp. 151-164.

[K98] Keller, G.: *Equilibrium states in ergodic theory.* London Mathematics Society Student Texts.Cambridge Universtity Press 1998.

[LS82] Ledrappier, L.; Strelcyn, J-M.: *A proof of the estimation from below in Pesin's formula.* Ergod. Th. & Dynam. Sys. **2** 1982 pp. 203-219.

[LY85] Ledrappier, L.; Young, L.-S.: *The metric entropy of diffeomorphisms Part I: Characterization of measures satisfying Pesin's entropy formula.* Ann. of Math. **122** 1985 pp. 509-539.

[M81] Mañé, R.: *A Proof of Pesin's Formula.* Ergod. Th. & Dynam. Sys. 1981 **1**, 95-102 and *Errata to 'A Proof of Pesin's Formula'.* Ergod. Th. & Dynam. Sys. 1983 **3**, 159-160.

[P77] Pesin, Ya.: *Characteristic Lyapunov exponents and smooth ergodic theory.* Russ. Math. Surv. **32:4** 1977 pp. 55-114.

[P84] Pugh, C.: *The $C^{1+\alpha}$ hypothesis in Pesin theory.* Publ. Math. IHES, **59** 1984 pp. 143-161.

[Q11] Qiu, H.: *Existence and uniqueness of SRB measure on C1 generic hyperbolic attractors.* Commun. Math. Phys. **302** 2011 pp. 345-357.

[R62] Rohlin, V.: *On the fundamental ideas of measure theory,* A. M. S. Transl. **1** 1962 pp. 1-52.

[R78] Ruelle, D.: *An inequality for the entropy of differentiable maps.* Bol. Soc. Bras. Mat. **9** 1978 pp. 83-87.

[ST10] Sun, W.; Tian, X.: *Pesin set, closing lemma and shadowing lemma in C^1 non-uniformly hyperbolic systems with limit domination.* arXiv:1004.0486v1 [math.DS] 2010.

[ST12] Sun, W.; Tian, X.: *Dominated splitting and Pesin's entropy formula.* Disc. and Cont. Dyn. Syst. **32-4** 2012 pp. 1421-1434.

[T02] Tahzibi, A.: *C^1-generic Pesin's entropy formula.* C. R. Acad. Sci. Paris, Ser. I **335** 2002 pp. 1057-1062.